1. Use the repertoire method, from the textbook “Concrete Mathematics” by Graham, Knuth and Patashnik (2nd Edition, 1998 printing) to find a closed form expression for the following sum:

\[ \sum_{k=0}^{n} (-1)^k k^2. \]

Hint: Use the recurrences

\[
\begin{align*}
  g(0) &= \alpha, \\
  g(n) &= g(n-1) + (-1)^n (\beta + \gamma n + \delta n^2) \quad \text{for } n > 0,
\end{align*}
\]

with a repertoire chosen from among the functions of the form \((-1)^n f(n)\), where \(f(n)\) is a polynomial of low degree.

Reference: in GKP – Problem 2.13 (on page 63). Note: The problem is solved in the back of GKP but there seem to be four errors in the last two lines of the solution (three places where \((-1)^n\) was inserted erroneously and one place where it should have been inserted but was not).

2. Recall that the forward difference operator \(\Delta = E - 1\), where the shift operator \(E\) takes a function \(f\) to \(Ef : x \mapsto f(x+1)\). Expanding \((E-1)^n\) as a binomial power and applying the expanded expression to \(f\) gives formula (5.40) from GKP

\[ \Delta^\ell f(x) = \sum_k \binom{\ell}{k} (-1)^{\ell-k} f(x+k), \quad \ell \geq 0. \quad (5.40) \]

(a) Apply identity (5.40) to a suitably chosen function \(f(x)\) in order to deduce the following identity:

\[ (-1)^{\ell+m} \binom{x-m}{n-\ell} = \sum_k \binom{\ell}{k} (-1)^{m-k} \binom{x+k-m}{n}, \quad \text{integer } \ell \geq 0, \text{ integers } m, n. \quad (1) \]

(b) Show that formula (1) implies formula (5.24) in table 169, namely:

\[ \sum_k \binom{\ell}{m+k} \binom{s+k}{n} (-1)^k = (-1)^{\ell+m} \binom{s-m}{n-\ell} \quad (5.24) \]

3. Consider the binomial coefficient identity for any integer \(n \geq 0\)

\[ \sum_{k>n} \binom{n+k}{n} 2^{-k} = 2^n. \quad (2) \]

What is the hypergeometric form of this identity?
4. A possible alternative to the preceding. Convert the sum

\[ \sum_k \binom{n}{m+k} \binom{m+k}{2k} \]

to a hypergeometric series. (See Exercise 5.24)

5. Recall (5.83), Euler’s definition of factorials,

\[ \frac{1}{z!} = \lim_{n \to \infty} \binom{n+z}{n} n^{-z}. \] (5.83)

Show that this definition is consistent with the ordinary definition of factorials by showing that, when \( z \) is a positive integer \( m \), the limit in (5.83) is \( 1/m! \). (See Exercise 5.21)