Exercise IV

Remarks: In all the algorithms, always explain their correctness and analyze their complexity.

• Question 1: Let $G$ be a graph that looks as follows:

![Graph](image)

The graph has paths that are all disjoint except for the start point $s$ that is mutual to all paths. Let $t$ be the number of paths, and $k$ be the length of every path. Let $v$ be the endvertex of one of the paths. Compute $h_{s,v}$.

Answer: Note that all edges in the graph are bridges. The number of edges in the graph is $m = t(k - 1)$. Split the path from $s$ to $v$ into its edges, $(s, x)$ and then $(x, y)$ and so on, until the last edge $(z, v)$. As explained in class:

$$h_{s,v} = h_{s,x} + h_{x,y} + \ldots + h_{z,v}.$$  

Observe that because the edges are bridges, $h_{s,v} + h_{v,s} = 2m = 2t(k - 1)$, and the same for all other edges. So:

$$h_{s,v} + h_{v,s} = (h_{s,x} + h_{x,s}) + (h_{x,y} + h_{y,x}) + \ldots + (h_{z,v} + h_{v,z}) = (k - 1)\cdot 2m.$$  

But, we know that $h_{v,s} = (k - 1)^2$ (its simply as in a path). So

$$h_{s,v} = 2(k - 1)^2 \cdot t - (k - 1)^2.$$
**Question 2:** Let $G$ be a directed and strongly connected graph with maximum out-degree $\Delta$. Let $v$ and $u$ be two vertices in the graph.

1. Show that with probability at least $1/\Delta^{n-1}$, a random walk that starts at $v$ arrives to $u$ in at most $n-1$ steps.

**Answer:** As the graph is strongly connected, there is some simple path $P$ from $v$ to $u$. The length of $P$ is at most $n-1$ (as otherwise $P$ must contain a cycle). We can now compute the probability that the random walk crosses the graph exactly along the edges of $P$. Let $w$ be the current location of the random walk. In the next step, there is one “correct” edge (the one from $P$) among the (at most) $\Delta$ outgoing edges of $w$. The probability to continue over this edge is $1/\Delta$. The probability to do so repeatedly over all the edges of $P$ is $1/\Delta \cdot |P|$. As $|P| \leq n-1$, the claim follows.

2. Show that the hitting time $h_{v,u}$ (the expected time to get from $v$ to $u$) is finite in every directed graph.

**Answer:** Assume we take $\rho \cdot \Delta^{n-1}$ walks each of length $n-1$ ($\rho \cdot \Delta^{n-1} \cdot (n-1)$ length walk in total). By the above claim, the probability to hit $u$ in a single walk of length $n-1$ is at least $1/\Delta^{n-1}$. Thus, the probability not to hit $u$ in a single walk is at most $1 - 1/\Delta^{n-1}$. The probability not to hit $u$ in all the $\rho \cdot \Delta^{n-1}$ walks is

$$(1 - 1/\Delta^{n-1})^\rho < e^{-\rho}.$$ 

By definition, the hitting time $h_{v,u}$ is:

$$\sum_{i=1}^{\infty} i \cdot Pr( \text{It takes } i \text{ time units to get from } v \text{ to } u) =$$

$$\sum_{i=1}^{n-2} i \cdot Pr( \text{It takes } i \text{ time units to get from } v \text{ to } u) + \sum_{i=n-1}^{\infty} i \cdot Pr( \text{It takes } i \text{ time units to get from } v \text{ to } u)$$

The first sum of the 2 is clearly finite. We now bound:

$$\sum_{i=n-1}^{\infty} i \cdot Pr( \text{It takes } i \text{ time units to get from } v \text{ to } u) \leq \sum_{i=n-1}^{\infty} i \cdot exp(i/n - 1).$$

The last inequality was proven in the previous item. Obviously, for $i > (n-1)^2$, $i \cdot exp(i/n - 1) << 1/i^2$. Hence, we can bound the tail of the sum by $\sum_{i \geq (n-1)^2} 1/i^2$ which is a constant. Hence, the sum is finite.

**Question 3:** Consider the following directed strongly connected graph: The graph is essentially a directed path, with all vertices in the path having an edge going into the start vertex of the path. Say that the path has length $n$ and let $0$ and $n$ be the left and right endvertices of the path. Show that $h_{0,n} = \Omega(2^n)$.
Answer: A typical walk, is currently at some vertex $w$. The vertex $w$ has two choices (two outgoing edges). One edge, lets call it $f$, goes one step forward and the second edge $b$ goes back to beginning at 0. In order to cross from 0 to $n$, we need $n$ consecutive $f$ moves (any $b$ move brings us back to the beginning). The probability for an $f$ or $b$ move is $1/2$. We showed in the first exercise that the expected streak of $f$, (actually, the expected size of the largest consecutive series of either $f$ or $b$) in a sequence of $k$ coin tosses (when $f$ and $b$ occur with probability $1/2$) is $\log k$. We need $\log k = n$ to get $n$ consecutive $f$. So, we need $k = 2^n$ steps.

• Question 4: Let $G$ be a graph with $\text{deg}(v) \geq 2n/3$ for every $v$.

1. Show that the neighbors of every two vertices have large intersection: $|N(u) \cap N(v)| \geq n/3$ for very $u, v \in V$.

   Answer: Note that $|N(u)| + |N(v)| \geq 4n/3$. As $n \geq |N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)|$ we get that $-|N(u) \cap N(v)| \geq n/3$.

2. Say that the move is now at $w$. Let $v$ be another vertex. Show that with probability at least $1/3$ the next vertex in the walk is a neighbor of $v$.

   Answer: Since $|N(w) \cap N(v)| \geq n/3$, and $|N(w)| \leq n$, we get that there are at least a third of the neighbors of $w$ that are neighbors of $v$ as well (namely,
\[ |N(v) \cap N(w)|/|N(w)| \geq 1/3. \] Since the next walk chooses a random vertex, the claim follows.

3. Show that \( h_{u,v} \) is \( O(n) \) (for any \( u \) and \( v \)) in such a graph.

**Answer:** With probability at least \( 1/3 \), we get to \( v \) in the next two moves: go to a neighbor of \( v \) (probability at least \( 1/3 \)) and then go to \( v \) (probability at least \( 1/n \)). Thus, we can treat reaching \( v \) as a geometric variable with probability of success at least \( 1/3n \). Thus, the expected number of steps to get to \( v \) is at most \( 2 \cdot 3n \) (we have multiplied by 2, because it takes two steps to get the \( 1/3n \) probability of getting to \( v \)).

**Question 5:** Show that the cover time of the graph in question 4 is \( O(n \cdot \log n) \).

**Answer:** We know that with probability \( 1/3n \) or more, we reach a vertex \( v \) (regardless of the position of the walk right now). Say we take \( 6 \cdot \rho \cdot n \) length walk. The walk can be regarded as \( 3\rho \cdot n \) independent 2 step walks. The probability to get to \( v \) in a specific 2 step stage is at least \( 1/3n \). Thus, the probability not to get to \( v \) in a two step walk is at most \( 1 - 1/3n \). The probability not to get to \( v \) at \( 6 \cdot \rho \cdot n \) length walk is at most

\[
\left(1 - \frac{1}{3n}\right)^{3\rho n} = \exp(-\rho).
\]

The probability that there is at least one vertex that we have not reached after a \( 6 \cdot \rho \cdot n \) walk is bounded as follows:

\[
Pr(\text{ We did not cover all the graph }) = Pr(\bigcup_{v \in V} \text{ We did not cover } v) < \sum_{v \in V} Pr(\text{ We did not cover } v) < n \cdot \exp(-\rho).
\]

For any \( i \), put \( i = 6 \cdot \rho \cdot n \), or \( \rho = i/6n \). Thus, the probability that the cover time is at least \( i = 6\rho \) is at most \( n \cdot \exp(-i/6n) \). Now, consider the cover time:

\[
\sum_{i=1}^{\infty} i \cdot Pr(\text{ The graph is covered after } i \text{ steps}).
\]

For \( 30 \log n \cdot n \leq i \leq n^2 \), the term \( i \cdot n \cdot \exp(-i/6n) \) is bounded by \( n^2 \cdot n \cdot n^{-5} \leq 1/n^2 \). For \( i > n^2 \) the term \( i \cdot n \cdot \exp(-i/6n) \) is bounded by \( 1/n^2 \). It follows that up to an additive constant, the cover time is:

\[
\sum_{i=1}^{30 \cdot n \cdot \log n} i \cdot Pr(\text{ The graph is covered after } i \text{ steps}) \leq 30 \cdot n \cdot \log n.
\]