Exercise I

Remarks: All the graphs here are without self loops and parallel edges, and anti-parallel edges. When we speak of a flow network, we mean there are capacities $c(e) \geq 0$ on the edges, the graph $G$ is directed with a source $s$ and a destination $t$. In all the algorithms, always explain their correctness and analyze their complexity. The complexity should be as small as possible. A correct algorithm with large complexity, may not get full credit.

• Question 1:
  1. Let $G$ be a flow network. Suppose we are given a legal flow function $f : E \rightarrow \mathcal{R}^+$, obeying the flow properties. Design an algorithm that checks if $f$ is a maximum flow.
  2. Let $G(V_1, V_2, E)$ be a bipartite graph. Suppose we are given a matching $M \subseteq E$. Design an algorithm that checks if $M$ is a maximum matching in $G$.

• Question 2: A matching $M$ is called maximal if $M$ is not strictly contained in another matching, namely, if there is no $M' \neq M$ such that $M \subseteq M'$.
  1. Design an algorithm that finds a maximal matching in an arbitrary undirected graph.
  2. Let $M$ be a maximal matching and $M^*$ be a maximum matching (in an arbitrary graph). Show that $|M| \leq |M^*| \leq 2 \cdot |M|$.

• Question 3: Let $M$ be a matching in an arbitrary graph. Prove that there is a maximum matching $M^*$, that contains all of $V(M)$ (namely, every vertex that was matched in $M$ is also matched in $M^*$).

• Question 4: Let $G$ be an undirected and connected graph. An edge-cut $E' \subseteq E$, is a set of edges such that $G'(V, E \setminus E')$ is disconnected. Namely, if you remove the edges of $E'$ from $G$, the graph turns disconnected. Design an algorithm that finds an edge-cut $E'$ of minimum size.

• Question 5: Suppose we are given a graph $G$ and also a maximum flow function $f$. Suppose the capacity of an edge $e$ is increased by 1. Design an algorithm that updates $f$ to a new maximum flow.