Home Exam

Remarks: All the graphs here are without self loops and parallel edges. In all the algorithms, always explain their correctness and analyze their complexity. The complexity should be as small as possible. A correct algorithm with large complexity, may not get full credit. YOU MAY ONLY USE WHAT WAS TAUGHT IN CLASS. NO FURTHER MATERIAL, LIKE DFS FOR DIRECTED GRAPHS CAN BE USED.

Choose 5 out of the next 6 questions.

Question 1: Let $G$ be an undirected graph. Write an algorithm that returns the set of all vertices whose degrees are strictly smaller than the degrees of all their neighbors.

Question 2: Let $T(V,E_1)$ and $T(V,E_2)$ be two different minimum spanning trees of the same weighted graph $G$. Show that if we make a vector of the weights of the edges appearing in $T_1$, by non-decreasing order, and do the same for $T_2$ the same vector results. Namely, the two trees have the same “weights vector”.

Question 3: The diameter in a tree is the length of the longest path between any two vertices in the tree. Namely, if $\text{dist}(u,v) = \max_{z,w}\{\text{dist}(z,w)\}$ then $\text{dist}(u,v)$ is the diameter. Give an algorithm that computes the diameter of a tree.

Question 4: Let $T(V,E)$ be a tree and $\mathcal{U}$ be a set of $2 \cdot k$ vertices in $T$ (namely, there is an even number of vertices in $\mathcal{U}$.) A pairing of $\mathcal{U}$ is a division of $\mathcal{U}$ into $k$ vertex disjoint pairs $(u_i,v_i)$ (namely, each vertex in $\mathcal{U}$ appears in exactly one pair). The corresponding path $\mathcal{P}_i$ of a $(u_i,v_i)$ pair is the (unique) simple path joining $u_i$ and $v_i$ in $T$.

1. Show that for any such $\mathcal{U}$ there is a pairing so that all the $\mathcal{P}_i$ are edge disjoint (but may not be vertex disjoint).
2. Give an algorithm that finds such a pairing.

Question 5: Let $G$ be an undirected graph. We say that two edges $e_1$ and $e_2$ are similar if and only if there is a simple cycle $\mathcal{C}$ containing both $e_1$ and $e_2$.

- Show that if $e_1$ and $e_2$ are similar, and so are $e_2$ and $e_3$, then so are $e_1$ and $e_3$.
- Let $S(e) = \{e' \mid e'$ is similar to $e$\}. Show that for every two edges, either $S(e_1) = S(e_2)$ or $S(e_1) \cap S(e_2) = \emptyset$.
- Show that the $S(e)$ sets partition the graph exactly to the bi-connected components.

Question 6: For the following graph, write for every vertex its $k(v)$ in a legal DFS run and its resulting $\text{Low}(v)$. In addition, describe the bi-connected components, and the super-graph. Finally, show by which order are the biconnected components going to be discovered.
Figure 1: A graph