Home Exam

Remarks: All the graphs here are without self loops and parallel edges. We use $|V| = n$. In all the algorithms, always explain their correctness and analyze their complexity. The complexity should be as small as possible. A correct algorithm with large complexity, may not get full credit.

Choose 5 out of the next 6 questions.

Question 1: We are given a set $O = \{o_1, \ldots, o_k\}$ of operators, and a set $S = \{s_1, \ldots, s_t\}$ of soldiers. Any operator knows only a subset of the soldiers. Let $B(O, S, E)$ be the bipartite graph in which we join the $s_i$ and $o_j$ by an edge, if they know each other. The operators need to deliver the soldiers a message, by calling over the phone with the following rules:

1. Each operator can call only soldiers that he/she knows.
2. Each call takes one unit of time.
3. Of course, because its a telephone, each operator can call one soldier at the time (so 4 calls, for example take 4 time units).

Give an algorithm that finds the minimum integer $p$ so that its possible to give the message to all the soldiers in $p$ time units.

Question 2: Suppose that we are given a procedure $A$ that gets as input a graph $G$ with positive or negative labels $\ell(e)$ over the edges. The procedure finds in $O(n)$ time the most negative cycle in $G$, namely, the cycle with cost $\min_C \ell(C)$. Explain how to use this procedure to find a maximum min-cost max-flow. Analyze the running time of the procedure.

Question 3: Given $2n$ numbers, $d_{\text{out}}^1, \ldots, d_{\text{out}}^n$, and $d_{\text{in}}^1, \ldots, d_{\text{in}}^n$, give an algorithm (using flow) that checks if there is a directed graph with the in and out degree of $v_i$ being $d_{\text{in}}^i$ and $d_{\text{out}}^i$.

Question 4: Let $G(V, E)$ be an undirected graph with a weight $w(e) > 0$ for all $e \in E$. For a set of vertices $V' \subseteq V$, let $E(V')$ be the set of edges with both vertices in $V'$, namely,

$$E(V') = \{e = (u, v) \in E \text{ such that } u, v \in V'\}.$$ 

Let $w(V') = \sum_{e \in E(V')} w(e)$. Let the density of $V'$ be defined as

$$\rho(V') = \frac{w(V')}{|V'|}.$$
Let $\rho^*$ be the maximum density, namely, $\rho^* = \max_{V' \subseteq V} \{\rho(V)\}$. The densest subgraph problem is to find the subset $V'$ with maximum density. Give a ratio 2 approximation algorithm for the problem, namely, an algorithm that finds a $V'$ so that $\rho(V') \geq \rho^*/2$. The algorithm should run in time $O(|E| \log |V|)$.

**Question 5:** Let $G$ be as in question 5. Let $k$ be a positive number.

1. Use flow to find a subset $V'$ that maximizes $w(V') - k|V'|$.
2. Give an algorithm that finds a subset $V'$ with maximum density

**Question 6:** Consider a set-cover instance with all degrees of $v_2 \in V_2$ at most $d$, namely, $\max_{v_2 \in V_2} \{\deg(v_2)\} = d$. Give a $d$ ratio approximation for the set cover problem in this case.