Math 356—Test I

March 9, 2009

Your Name:

Solve exactly six of the following eight problems. If you solve more than six problems, you must indicate which ones are to be graded.

(1) Using Pascal's rule and mathematical induction principle to prove that for any positive integer $n \geq 2$,

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} \cdots + \binom{n}{2} = \binom{n+1}{3}.$$

(2) Let a and b be integers such that $\gcd(a, b) = 1$. Show the following:
(a) $\gcd(2a + b, a + 2b) = 1$ or 3.
(b) $\gcd(a, b^2) = 1$.

(3) Find integers x, y, and z such that

$$\gcd(84, 180, 450) = 84x + 180y + 450z.$$

(4) A grocer bought apples and oranges at a total cost of \$8.39. If apples cost him 25 cents and oranges 18 cents each, how many of each type of the fruits did he buy?

(5) Show that for any prime p, $\sqrt[3]{p}$ is irrational.

(6) Let n be any integer greater than 3. Show that n, $n + 2$, and $n + 4$ cannot all be primes. (Hint: Write n in the forms of $6q + r$, $0 \leq r \leq 5$.)

(7) (a) Find the remainder of 41^{1000} divided by 7. (b) Find the remainder of the following number divided by 3:

$$1^4 + 2^4 + 3^4 + \ldots + 99^4.$$

(8) Show that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

TA1

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(1) Initial Step: For $n=2$, the left hand side is $\binom{2}{2}=1$ and the right hand side is $\binom{3}{3}=1$. Thus they are the same.

3 pts

Inductive Step: Assume that the identity holds for $n=k$. Then

$$\binom{2}{2}+\binom{3}{2}+\cdots+\binom{k}{2}=\binom{k+1}{3}.$$

Adding $\binom{k+1}{2}$ to both side,

$$\binom{2}{2}+\binom{3}{2}+\cdots+\binom{k}{2}+\binom{k+1}{2}=\binom{k+1}{3}+\binom{k+1}{2}.$$

$$=\binom{k+2}{3},$$

where the last equality follows from the Pascal's identity.

6 pts

Hence the equality hold for $n=k+1$. By the math induction principle, it holds for all $n \geq 2$. 1 pts

(2) (a) Since $3a = 2(2a+b)-(a+2b)$ and

$$3b = -(2a+b)+2(a+2b),$$

we have $d \mid 3a$ and $d \mid 3b$, where

$d = \gcd(2a+b, a+2b)$. Hence

$d \mid \gcd(3a, 3b)$. However,

$\gcd(3a, 3b) = 3\gcd(a,b) = 3$.

Hence $d=1$ or $d=3$.

TA2

(b) Since $\gcd(a,b)=1$, $ax+by=1$ for some $x,y\in\mathbb{Z}$.

Squaring both sides, we have

$$a^2x^2+2abxy+b^2y^2=1.$$

Hence $a(ax^2+2bxy)+b^2y^2=1$.

Therefore $au+b^2v=1$, where $u=ax^2+2bxy$, $v=y^2$.

Thus, $\gcd(a,b^2)=1$.

(3) Let $d=\gcd(84,180)$.

$$\begin{array}{r} 2 \\ 84\overline{)180} \\ 168 \\ \hline 12 \end{array} \qquad \begin{array}{r} 7 \\ 12\overline{)84} \\ 84 \\ \hline 0 \end{array}$$

$$d=12=(-2)\cdot 84+180$$ 3 pts

Let $\tilde{d}=\gcd(84,180,450)=\gcd(\gcd(84,180),450)$
$=\gcd(12,450)$.

$$\begin{array}{r} 37 \\ 12\overline{)450} \\ 36 \\ \hline 90 \\ 84 \\ \hline 6 \end{array} \qquad \begin{array}{r} 2 \\ 6\overline{)12} \\ 12 \\ \hline 0 \end{array}$$

$$\tilde{d}=6=(-37)\cdot 12+450$$ 3 pts

$$=(-37)((-2)84+180)+450$$

$$=74\cdot 84+(-37)\cdot 180+450$$ 4 pts

$$=84x+180y+450z, \text{ where } x=74,\ y=-37,\ z=1.$$

TA3

(4) Let x = # of apples; y = # of oranges.

Then
$$25x + 18y = 839 \qquad 3 \text{ pts}$$

Note that $\gcd(25, 18) = 1$.

$$18\,\overline{)\,25}^{\,1} \quad 25 - 18 = 7 \qquad 7\,\overline{)\,18}^{\,2} \quad 18 - 14 = 4 \qquad 4\,\overline{)\,7}^{\,1} \quad 7 - 4 = 3 \qquad 3\,\overline{)\,4}^{\,1} \quad 4 - 3 = 1$$

$$\begin{aligned}
1 &= (-1)\cdot 3 + 4 = (-1)[(-1)\cdot 4 + 7] + 4 \\
&= 2\cdot 4 + (-1)\cdot 7 = 2\cdot((-2)\cdot 7 + 18) + (-1)\cdot 7 \\
&= (-5)\cdot 7 + 2\cdot 18 \\
&= (-5)((-1)\cdot 18 + 25) + 2\cdot 18 \\
&= (-5)\cdot 25 + 7\cdot 18 \qquad 2 \text{ pts}
\end{aligned}$$

Hence $839 = ((-5)\cdot 839)\cdot 25 + (7\cdot 839)\cdot 18$

Therefore $x = -5\cdot 839$; $y = 7\cdot 839$ is a special solution. The general solution is:

$$x = \underbrace{-5\cdot 839}_{=4195} + 18t; \quad y = \underbrace{7\cdot 839}_{5873} + 25t$$

2 pts

$$x > 0 \Rightarrow t > \frac{5\cdot 839}{18} \approx 233.1$$

$$y > 0 \Rightarrow t < \frac{7\cdot 839}{25} \approx 234.9$$

Hence $t = 234$. Hence

$x = 17$, $y = 23$. 3 pts

The grocer bought 17 apples & 23 oranges.

TA4

(5) Proving by contradiction, we assume that $\sqrt[3]{p}$ is rational. Then $\sqrt[3]{p} = \frac{a}{b}$, where a, b are positive integers such that $\gcd(a, b) = 1$.

2pts

Then $p = (\frac{a}{b})^3 = \frac{a^3}{b^3} \Rightarrow a^3 = p \cdot b^3$.

Since $p | a^3$, we know that $p | a$ by Euclid's Lemma. Hence $a = p \cdot k$ for some positive integer k. Thus,

$$(pk)^3 = p \cdot b^3 \Rightarrow p^2 k^3 = b^3.$$

Hence $p | b^3$. Therefore, $p | b$.

Since $p | a$ and $p | b$, we have $p | \gcd(a, b)$, which contradicts the assumption that $\gcd(a, b) = 1$.

(6) By the Division algorithm, every integer is of the forms of $6q + r$, $0 \le r \le 5$.

If n is a prime ≥ 3, then n cannot be of form: $6q$, $6q+2$, $6q+3$.

4 pts

If $n = 6q+1$, then $n+2 = 6q+3 = 3(2q+1)$ is not a prime.

3 pts

If $n = 6q+5$, then $n+4 = 6q+9 = 3(2q+3)$ is

TA5

not a prime. Hence n, $n+2$, $n+4$ cannot be all primes. 3 pts

(7) (a)

$$41 \equiv (-1) \bmod 7$$

Hence $41^{1000} \equiv (-1)^{1000} \bmod 7 \equiv 1 \bmod 7$.

Thus 41^{1000} divided by 7 has remainder 1. 5 pts

(b)

$$3k \equiv 0 \bmod 3 \Rightarrow (3k)^4 \equiv 0 \bmod 3$$

$$(3k+1) \equiv 1 \bmod 3 \Rightarrow (3k+1)^4 \equiv 1 \bmod 3$$

$$(3k+2) \equiv 2 \bmod 3 \Rightarrow (3k+2)^4 \equiv 2^4 \bmod 3 \equiv 1 \bmod 3$$

Since there are exactly 33 numbers between 1 and 99 of each of the forms $3k$, $3k+1$, $3k+2$,

$$1^4 + 2^4 + 3^4 + \cdots + 99^4 \equiv 33\,(0+1+1) \bmod 3$$

$$\equiv 0 \bmod 3 \quad 5 \text{ pts}$$

(8) Let $N = a_0 + a_1 \cdot 10 + \cdots + a_m \cdot 10^m$, $0 \le a_i \le 9$. 2 pts

Since

$$1 \equiv 1 \bmod 3 \Rightarrow a_0 \equiv a_0 \bmod 3$$

$$10 \equiv 1 \bmod 3 \Rightarrow a_1 \cdot 10 \equiv a_1 \bmod 3$$

$$10^2 \equiv 1 \bmod 3 \Rightarrow a_2 \cdot 10^2 \equiv a_2 \bmod 3$$

$$\cdots$$

$$10^m \equiv 1 \bmod 3 \Rightarrow a_m \cdot 10^m \equiv a_m \bmod 3,$$

6 pts

We have

$$N = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_m 10^m \quad 2 \text{ pts}$$

$$\equiv (a_0 + a_1 + \cdots + a_m) \bmod 3.$$

Hence $3 \mid N \iff 3 \mid (a_0 + a_1 + \cdots + a_m)$.