p1

H.W. #4 Solutions

§6.1: 5, 9; §6.2: 3, 4 a) b), §6.3: 5, 6; §7.2: 5, 8

§6.1 #5: We will use the formulas: If $n = p_1^{k_1} \cdots p_r^{k_r}$ is the prime factorization of n, then

$$\tau(n) = (k_1+1)\cdots(k_r+1)$$

and

$$\sigma(n) = (1+p_1+\cdots+p_1^{k_1})\cdots(1+p_r+\cdots+p_r^{k_r}).$$

(a)
$$\tau(3655) = \tau(5\cdot 7\cdot 13) = 2\cdot 2\cdot 2 = 8$$
$$\tau(3656) = \tau(2^3\cdot 457) = (3+1)\cdot 2 = 8$$
$$\tau(3657) = \tau(3\cdot 23\cdot 53) = 2\cdot 2\cdot 2 = 8$$
$$\tau(3658) = \tau(2\cdot 3\cdot 59) = 2\cdot 2\cdot 2 = 8$$

and
$$\tau(4503) = \tau(3\cdot 19\cdot 73) = 8$$
$$\tau(4504) = \tau(2^3\cdot 563) = 4\cdot 2 = 8$$
$$\tau(4505) = \tau(5\cdot 17\cdot 53) = 2\cdot 2\cdot 2 = 8$$
$$\tau(4506) = \tau(2\cdot 3\cdot 751) = 2\cdot 2\cdot 2 = 8$$

(b)
$$\sigma(14) = \sigma(2\cdot 7) = (1+2)(1+7) = 24$$
$$\sigma(15) = \sigma(3\cdot 5) = (1+3)(1+5) = 24$$

&
$$\sigma(206) = \sigma(2\cdot 103) = (1+2)(1+103) = 312$$
$$\sigma(207) = \sigma(3^2\cdot 23) = (1+3+3^2)(1+23) = 312$$

&
$$\sigma(957) = \sigma(3\cdot 11\cdot 29) = (1+3)(1+11)(1+29) = 1440$$
$$\sigma(958) = \sigma(2\cdot 479) = (1+2)(1+479) = 1440$$

[redacted]

D2

§6.1 #9

Since n is square-free, the prime factorization of n has the form

$$n = p_1 \cdots p_r$$

where $p_1, \ldots, p_r$ are distinct prime factors.

In this case

$$\tau(n) = \underbrace{(1+1) \cdots (1+1)}_{r\text{-times}} = 2^r.$$

§6.2 #3

Since $\mu(d)$ is multiplicative, and by assumption $f(d)$ is also multiplicative. Then $\mu(d) \cdot f(d)$ is multiplicative. Let

$$F(n) = \sum_{d \mid n} \mu(d) f(d).$$

Then $F(n)$ is also multiplicative (Thm 6.4).

Assume that $n = p_1^{k_1} \cdots p_r^{k_r}$ is the prime factorization of n. Then

$$F(n) = F(p_1^{k_1}) \cdots F(p_r^{k_r}).$$

However, by the definition of $F(n)$, we have

D3

$$F(p_i^{k_i}) = \overset{=1}{\mu(1)} f(1) + \overset{=-1}{\mu(p_i)} f(p_i) + \overset{=0}{\mu(p_i^2)} f(p_i^2) + \cdots + \underset{=0}{\mu(p_i^{k_i})} f(p_i^{k_i})$$

$$= f(1) - f(p_i) = 1 - f(p_i), \quad 1 \le i \le r.$$

(We know $f(1) = f(1 \cdot 1) = f(1) \cdot f(1) \Rightarrow f(1) = 1$, because f is not identically zero.)

Therefore $F(n) = (1 - f(p_1)) \cdots (1 - f(p_r))$.

§6.2 #4 We use the formula in #3.

a) In this case, $f(d) = \tau(d)$; $f(p_i) = 2$.

Hence

$$\sum_{d|n} \mu(d)\tau(d) = \underbrace{(1-2)\cdots(1-2)}_{r\text{-times}} = (-1)^r.$$

b) In this case, $f(d) = \sigma(d)$; $f(p_i) = \tau(p_i) = (1+p_i)$

Hence

$$\sum_{d|n} \mu(d)\sigma(d) = (1-(1+p_1)) \cdots (1-(1+p_r))$$

$$= (-p_1) \cdots (-p_r) = (-1)^r p_1 \cdots p_r.$$

§6.3 #5

(a) The exponent of 2 in 1000! is

$$\sum_{k=1}^{\infty} \left[\frac{1000}{2^k}\right] = \left[\frac{1000}{2}\right] + \left[\frac{1000}{2^2}\right] + \left[\frac{1000}{2^3}\right] + \cdots + \left[\frac{1000}{2^9}\right]$$

$$= 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1$$

$$= 994.$$

The exponent of 5 in 1000! is:

(D4)

$$\sum_{k=1}^{\infty}\left[\frac{1000}{5^k}\right] = \left[\frac{1000}{5}\right]+\left[\frac{1000}{5^2}\right]+\left[\frac{1000}{5^3}\right]+\left[\frac{1000}{5^4}\right]$$

$$= 200+40+8+1 = 249.$$

Therefore the # of zeroes in 1000! is 249:

$$2^{249}\cdot 5^{249} = (10)^{249}.$$

(b) The # of zeroes in $n!$ is at least

$\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]$. We solve

$$\frac{n}{5}+\frac{n}{5^2}+\frac{n}{5^3} = \frac{31}{125}n \approx 37$$

$\Rightarrow n \approx 149$. Now we use trial and error.

By trial and error, we know that for ~~these~~ values of n, $n!$ has exactly 37 zeroes:

X $n=149$ $\left[\frac{149}{5}\right]+\left[\frac{149}{5^2}\right]+\left[\frac{149}{5^3}\right] = 29+5+1=35$

V $n=150$: $\left[\frac{150}{5}\right]+\left[\frac{150}{5^2}\right]+\left[\frac{150}{5^3}\right] = 30+6+1=37$

V $n=151$: $\left[\frac{151}{5}\right]+\left[\frac{151}{5^2}\right]+\left[\frac{151}{5^3}\right] = 30+6+1=37$

V $n=152$: $\left[\frac{152}{5}\right]+\left[\frac{152}{5^2}\right]+\left[\frac{152}{5^3}\right] = 30+6+1=37$

V $n=153$: $\left[\frac{153}{5}\right]+\left[\frac{153}{5^2}\right]+\left[\frac{153}{5^3}\right] = 30+6+1=37$

V $n=154$: $\left[\frac{154}{5}\right]+\left[\frac{154}{5^2}\right]+\left[\frac{154}{5^3}\right] = 30+6+1=37$

X $n=155$ $\left[\frac{155}{5}\right]+\left[\frac{155}{5^2}\right]+\left[\frac{155}{5^3}\right] = 31+6+1=38$

DS

§6.3 #6

(a) $\frac{(2n)!}{(n!)^2} = \frac{\cancel{1\cdot 2 \cdots n}\cdot(n+1)\cdots(2n)}{\cancel{1\cdot 2\cdots n}\cdot 1\cdot 2\cdots n}$

$= \frac{2(n+1)\cdots(2n-1)}{1\cdot 2\cdots(n-1)}$

$= 2\cdot\frac{(2n-1)!}{(n-1)!\,n!} = 2\binom{2n-1}{n-1}.$

Since $\binom{m}{r}$ is an integers for any positive integer $m \geq r$, hence $\frac{(2n)!}{(n!)^2}$ is even.

(b) The highest power of p in $(2n)!$ is $\sum_{k=1}^{\infty}\left[\frac{2n}{p^k}\right]$ and in $(n!)^2$ is $2\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$.

Hence in $\frac{(2n)!}{(n!)^2}$ is $\sum_{k=1}^{\infty}\left(\left[\frac{2n}{p^k}\right]-2\left[\frac{n}{p^k}\right]\right)$.

c) If $n<p<2n$, then

$\left[\frac{n}{p}\right]=0, \quad \left[\frac{n}{p^k}\right]=0, \quad k\geq 2$

$\left[\frac{2n}{p}\right]=1, \quad \left[\frac{2n}{p^k}\right]=0, \quad k\geq 2.$

Hence the power of p in $\frac{(2n)!}{(n!)^2}$ is

$1-2\cdot 0 = 1.$

D6

§7.2 #5 We use the formula: $n = p_1^{k_1} \cdots p_r^{k_r}$ (prime factorization), then

$$\phi(n) = (p_1^{k_1} - p_1^{k_1-1}) \cdots (p_r^{k_1} - p_r^{k_r-1}).$$

when $n = 2(2p-1)$, where both p & $2p-1$ are odd primes,

$$\phi(n) = (2-1)((2p-1)-1) = 2(p-1).$$

$$\phi(n+2) = \phi(2(2p-1)+2) = \phi(2^2 \cdot p) = (2^2-2)(p-1)$$
$$= 2(p-1).$$

Thus $\phi(n) = \phi(n+2)$.

§7.2 #8 If $n = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \cdots, p_r$ are distinct odd primes, then

$$\phi(n) = p_1^{k_1-1}(p_1-1) \cdots p_r^{k_r-1}(p_r-1).$$

Since each $(p_i - 1)$ is even, is of the form of $p_i - 1 = 2m_i$. Hence

$$\phi(n) = p_1^{k_1-1} \cdots p_r^{k_r-1} \cdot m_1 \cdots m_r \cdot 2^r.$$

Thus $2^r \mid \phi(n)$.

D7

Solutions to selected problems in §7.3 & 7.4

§7.3: 2,5,9; §7.4: 1,5,6

§7.3 #2:

Since $51 = 3\cdot 17$, $\gcd(51,10)=1$, $\phi(51) = \phi(3)\phi(17) = 2\times 16 = 32$,

by Euler's Theorem,

$$10^{\phi(51)} \equiv 10^{32} \equiv 1 \pmod{51}.$$

Hence $10^{32n} \equiv 1 \pmod{51}$.

On the other hand,

$$10^2 \equiv -2 \bmod 51,$$

$$\Rightarrow 10^8 = (10^2)^4 \equiv (-2)^4 \equiv 16 \pmod{51}.$$

Hence $10^9 \equiv 160 \pmod{51} \equiv 7 \bmod 51$.

Hence $10^{32n+9} \equiv 7 \bmod 51$.

Therefore $51 \mid 10^{32n+9} - 7$

§7.3 #5: Since $\gcd(m,n)=1$,

$m^{\phi(n)} \equiv 1 \bmod n \Rightarrow n \mid m^{\phi(n)} - 1 \Rightarrow n \mid n^{\phi(m)} + m^{\phi(n)} - 1$

$n^{\phi(m)} \equiv 1 \bmod m \Rightarrow m \mid n^{\phi(n)} - 1 \Rightarrow m \mid m^{\phi(n)} + n^{\phi(m)} - 1$

(since $n \mid n^{\phi(m)}$; $m \mid m^{\phi(n)}$).

Therefore, $m\cdot n \mid m^{\phi(n)} + n^{\phi(m)} - 1$.

Hence $m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$.

D8

§7.3 #9 $\Phi(77) = \Phi(7\cdot 11) = 6\cdot 10 = 60$, $2^{60} \equiv 1 \pmod{77}$

$$\Rightarrow 2^{100000} \equiv 2^{60\times 1666+40} \equiv (2^{60})^{1666}\cdot 2^{40}$$

$$\equiv 2^{40} \pmod{77}.$$

We have $2^{10} \equiv 1024 \equiv 23 \mod 77$,

$2^{20} \equiv 529 \equiv -10 \mod 77$

$2^{40} \equiv 100 \mod 77 \equiv 23 \mod 77$.

Hence $2^{100000} \equiv 23 \mod 77$.

§7.4 #1 Any positive integer n can be written as $n = 2^k\cdot N$, where N is an odd integer and $k \geq 0$. The divisor of n are either of form $2^k\cdot d$ where $d \mid N$ or is a divisor of $2^{k-1}\cdot N$. Hence

$$\sum_{d\mid n} (-1)^{\frac{n}{d}} \Phi(d) = \sum_{d\mid N} (-1)^{\frac{2^k\cdot N}{2^k\cdot d}} \Phi(2^k\cdot d) + \sum_{d\mid 2^{k-1}\cdot N} (-1)^{\frac{2^k\cdot N}{d}} \Phi(d).$$

Since N is odd, $\frac{2^k\cdot N}{2^k\cdot d} = \frac{N}{d}$ is also odd.

Hence $$\sum_{d\mid N} (-1)^{\frac{2^k N}{2^k\cdot d}} \Phi(2^k d) = -\sum_{d\mid N} \Phi(2^k d)$$

$$= -\sum_{d\mid N} \Phi(2^k)\,\Phi(d) = -(2^k - 2^{k-1}) \sum_{d\mid N} \Phi(d).$$

D9

$$= -2^{k-1}(2-1)\sum_{d|N}\Phi(d) = -2^{k-1}N.$$

On the other hand, $\frac{2^k\cdot N}{d}$ is even if $d\,|\,2^{k-1}\cdot N$.

Hence

$$\sum_{d|2^{k-1}N}(-1)^{\frac{2^k\cdot N}{d}}\Phi(d) = \sum_{d|2^{k-1}N}\Phi(d)$$

$$= 2^{k-1}\cdot N \quad \text{(We use Gauss' Thm : Thm 7.6)}$$

Therefore, when n is even ($k \geq 1$),

$$\sum_{d|n}(-1)^{\frac{n}{d}}\Phi(d) = 2^{k-1}\cdot N - 2^{k-1}\cdot N \equiv 0.$$

When n is odd (i.e., there is no second expression $\sum_{d|2^{k-1}N}(-1)^{\frac{2^kN}{d}}\Phi(d)$),

$$\sum_{d|n}(-1)^{\frac{n}{d}}\Phi(d) = -\sum_{d|n}\Phi(d) = -n. \quad \boxtimes$$

§7.4 #5.

The idea is that if $f(d)$ is multiplicative, then

$$F(n) = \sum_{d|n}f(d) \text{ is multiplicative, (Thm 6.8).}$$

For multiplicative function,

$$F(p_1^{k_1}\cdots p_r^{k_r}) = F(p_1^{k_1})\cdots F(p_r^{k_r}).$$

D10

Therefore, we need only to check the formula when $n = p^k$, p - prime.

(a) when $n = p^k$,

$$\sum_{d|n} \mu(d)\phi(d) = \overset{=1}{\mu(1)}\phi(1) + \overset{=-1}{\mu(p)}\phi(p) + \overset{=0}{\mu(p^2)}\phi(p^2) + \cdots + \underset{=0}{\mu(p^k)}\phi(p^k)$$

$$= \phi(1) - \phi(p) = 1 - (p-1) = 2 - p.$$

(b) when $n = p^k$,

$$\sum_{d|n} d\,\phi(d) = 1\cdot\phi(1) + p\cdot\phi(p) + \cdots + p^k\phi(p^k)$$

$$= 1 + p(p-1) + p^2(p^2-p) + \cdots + p^k(p^k - p^{k-1})$$

$$= 1 - p + p^2 - p^3 + p^4 + \cdots - p^{2k-1} + p^{2k}$$

$$= \frac{p^{2k+1}+1}{p+1}.$$

(c) when $n = p^k$,

$$\sum_{d|n} \frac{\phi(d)}{d} = 1\cdot\phi(1) + p^{-1}\phi(p) + p^{-2}\phi(p^2) + \cdots + p^{-k}\phi(p^k)$$

$$= 1 + p^{-1}(p - p^0) + p^{-2}(p^2 - p) + \cdots + p^{-k}(p^k - 1)$$

$$= (k+1) - kp^{-1} = 1 + \frac{k(p-1)}{p}.$$

§7.4 #6 By Thm 6.11, If $F(n) = \sum_{d|n} f(d)$

$$\sum_{k=1}^{N} f(k)\left[\frac{N}{k}\right] = \sum_{k=1}^{n} F(k).$$

D11

Therefore, Let $F(n) = \sum_{d|n} \phi(d) = n$

$$\sum_{d=1}^{n} \phi(d) \left[\frac{n}{d}\right] = \sum_{k=1}^{n} F(k) = 1+2+\cdots+n = \frac{n(n+1)}{2}.$$

∎