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Math 356 HW #3

§3.3: 9; §4.2: 1, 4; §4.3: 7, 23.

§3.3 #9

(a) Any integer n can be expressed as
$6k, 6k+1, 6k+2, 6k+3, 6k+4, 6k+5.$

If n is prime, n can only take the form of $6k+1$, $6k+5$.

If $n = 6k+1$, then $n+2 = 6k+3 = 3(2k+1)$ is composite.

If $n = 6k+5$, then $n+4 = 6k+9 = 3(2k+3)$ is composite.

Therefore it is impossible for all $n, n+2, n+4$ are primes.

§4.2 #1

(a) $a \equiv b \pmod{n} \Rightarrow a-b = nk$ for some $k \in \mathbb{Z}$.

$m \mid n \Rightarrow n = m\ell$ for some $\ell \in \mathbb{Z}$.

Hence $a-b = m\ell k \Rightarrow a \equiv b \pmod{m}$.

(b) $a \equiv b \pmod{n} \Rightarrow a-b = nk$, for some $k \in \mathbb{Z}$.

Hence $ac - bc = nck$. Therefore, $ac \equiv bc \pmod{nc}$.

(c) $a \equiv b \pmod{n} \Rightarrow a-b = nk$, for some $k \in \mathbb{Z}$.

Since a, b, n are all divisible by $d > 0$, we have
$\frac{a}{d} - \frac{b}{d} = \frac{n}{d}k.$

(2)

$$\frac{a}{d} \equiv \frac{b}{d} \pmod{\frac{n}{d}}.$$

§4.2 #4

(a) $2 = 2 \pmod 7$, $2^2 = 4 \pmod 7$

$2^3 = 1 \pmod 7$.

Since $50 = 3 \cdot 16 + 2$

$$2^{50} \equiv (2^3)^{16} \cdot 2^2 \equiv (1)^{16} \cdot 2^2 \pmod 7$$

$$\equiv 4 \pmod 7.$$

Hence the remainder of 2^{50} divided by 7 is 4.

Since $41 \equiv -1 \pmod 7$

$$41^{65} \equiv (-1)^{65} \pmod 7$$

$$\equiv -1 \pmod 7 \equiv 6 \pmod 7.$$

Hence the remainder of 41^{65} divided by 7 is 6.

(b) Every integer n is of one of the forms:

$4k, 4k+1, 4k+2, 4k+3$.

$4k \equiv 0 \pmod 4 \Rightarrow (4k)^5 \equiv 0^5 \pmod 4 \equiv 0 \pmod 4$

$4k+1 \equiv 1 \pmod 4 \Rightarrow (4k+1)^5 \equiv 1^5 \pmod 4 = 1 \pmod 4$

$4k+2 \equiv 2 \pmod 4 \Rightarrow (4k+2)^5 \equiv 2^5 \pmod 4 \equiv 0 \pmod 4$

$4k+3 \equiv 3 \pmod 4 \Rightarrow (4k+3)^5 \equiv 3^5 \pmod 4 \equiv 3 \pmod 4$.

From 1 to 100, there are 25 integers of each

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of the forms $4k$, $4k+1$, $4k+2$, $4k+3$. Hence

$$1^5 + 2^5 + 3^5 + \cdots + 100^5 \equiv 25(0+1+0+3) \pmod 4$$
$$\equiv 0 \mod 4.$$

Thus the remainder is 0.

§4.3 #7

(a) Let $N = a_m 10^m + a_{m-1} \cdot 10^{m-1} + \cdots + a_1 \cdot 10 + a_0$

$0 \le a_i \le 9$.

Since
$10 \equiv 0 \mod 2$ $\quad$ $a_0 = a_0 \mod 2$, $\quad a_1 \cdot 10 \equiv 0 \mod 2$
$10^2 \equiv 0 \mod 2$ $\quad$ $a_2 \cdot 10^2 \equiv 0 \mod 2$
$\cdots$ $\quad$ $\cdots$
$10^m \equiv 0 \mod 2$ $\quad$ $a_m \cdot 10^m \equiv 0 \mod 2$

$$\Rightarrow N \equiv a_m \cdot 10^m + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$$
$$\equiv a_0 \pmod 2.$$

Therefore $2 \mid N \iff 2 \mid a_0$, i.e., a_0 is even.

(b) Let $N = a_m \cdot 10^m + a_{m-1} \cdot 10^{m-1} + \cdots + a_1 \cdot 10 + a_0$

Since
$1 \equiv 1 \mod 3 \Rightarrow a_0 \equiv a_0 \pmod 3$
$10 \equiv 1 \mod 3 \Rightarrow a_1 \cdot 10 \equiv a_1 \pmod 3$
$\cdots$
$10^m = 1 \mod 3 \Rightarrow a_m \cdot 10^m \equiv a_m \pmod 3$

$$\Rightarrow N = a_m \cdot 10^m + \cdots + a_1 \cdot 10 + a_0$$
$$\equiv (a_m + \cdots + a_1 + a_0) \pmod 3$$

Hence $3 \mid N \iff 3 \mid a_m + \cdots + a_1 + a_0$.

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(c) $N = a_m \cdot 10^m + a_{m-1} \cdot 10^{m-1} + \cdots + a_1 \cdot 10 + a_0$

$1 \equiv 1 \pmod 4 \Rightarrow a_0 \equiv a_0 \pmod 4$

$10 \equiv 10 \pmod 4 \Rightarrow a_1 \cdot 10 \equiv a_1 \cdot 10 \pmod 4$

$10^2 \equiv 0 \pmod 4 \Rightarrow a_2 \cdot 10^2 \equiv 0 \pmod 4$

$\cdots\cdots$

$10^m \equiv 0 \pmod 4 \Rightarrow a_m \cdot 10^m \equiv 0 \pmod 4$

$\Rightarrow N \equiv (a_0 + a_1 \cdot 10 + 0 \cdots + 0) \bmod 4$

$\equiv (a_0 + a_1 \cdot 10) \bmod 4.$

Hence $4 \mid N \Leftrightarrow 4 \mid a_0 + a_1 \cdot 10.$

§4.3 #23 Since $72 = 8 \cdot 9$ and $72 \mid x67.9y$,

Using the fact that $8 \mid N \Leftrightarrow 8$ divides the last 3 digits of N (See the previous problem #7 for a proof of a similar fact), we have

$8 \mid 7.9y \Rightarrow y = 2.$

Since $9 \mid x67.92 \Rightarrow x + 6 + 7 + 9 + 2 \equiv 0 \bmod 9$

$\Rightarrow x \equiv -6 \pmod 9 \Rightarrow x \equiv 3 \pmod 9.$

Hence the receipt was: 367.92.