B1

Math 35 b H.W. #2

§2.4: 4 a) c), 13
§2.5: 6 8 b)
§3.1: 4 7
§3.2: 4, 13 a) b)
§3.3: 2, 26 a) b)

§2.4 #4 a) c)

(a) Let $d = \gcd(a+b, a-b)$.

Then $d \mid a+b$, $d \mid a-b$.

Since $2a = (a+b) + (a-b)$; $2b = (a+b) - (a-b)$,

we have $d \mid 2a$, $d \mid 2b$.

By Thm 2.7, $\gcd(2a, 2b) = 2\gcd(a,b) = 2$.

Hence $d \mid 2 \Rightarrow d = 1$ or $d = 2$.

(c) Let $d = \gcd(a+b, a^2+b^2)$.

Since $2a^2 = (a+b)(a-b) + (a^2+b^2)$;

$2b^2 = -(a+b)(a-b) + (a^2+b^2)$,

we have $d \mid 2a^2$, $d \mid 2b^2$.

By assumption, $\gcd(a,b) = 1$. Hence

$ax + by = 1$ for some $x, y \in \mathbb{Z}$.

Hence $ax = 1 - by$. Squaring both sides,

$a^2x^2 = 1 - 2by + b^2y^2$

$\Rightarrow$ $2by = 1 + b^2y^2 - a^2x^2$. squaring both sides again,

$4b^2y^2 = 1 + b^4y^4 + a^4x^4 - 2a^2x^2 + 2b^2y^2 - 2a^2b^2x^2y^2$.

Hence $-a^4x^4 + 2a^2x^2 + 2b^2y^2 - b^4y^4 + 2a^2b^2x^2y^2 = 1$

$\Rightarrow$ $a^2\underbrace{(-a^2x^4 + 2x^2)}_{u} + b^2\underbrace{(2y^2 - b^2y^4 + 2a^2x^2y^2)}_{v} = 1$

Hence $a^2u + b^2v = 1$. By Thm 2.4,

$\gcd(a^2, b^2) = 1$. (You can also apply Problem 20 a) in §2.3 twice to obtain this.)

Hence $\gcd(2a^2, 2b^2) = 2\gcd(a^2, b^2) = 2$. Therefore $d \mid 2 \Rightarrow d = 1$ or $d = 2$.

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(*) **§2.4 #12** Let $d = \gcd(198, 288)$. Then

$$\tilde{d} = \gcd(198, 288, 512) = \gcd(d, 512).$$

$$288 = 1 \cdot 198 + 90 \qquad 198 = 2 \cdot 90 + 18, \quad 18 = d \qquad 90 = 5 \cdot 18 + 0$$

$$d = 18 = 198 - 2 \cdot 90 = 198 - 2(288 - 198)$$
$$= 3 \cdot 198 + (-2) \cdot 288.$$

$$\tilde{d} = \gcd(18, 512).$$

$$512 = 28 \cdot 18 + 8 \qquad (512 - 36 = 152,\ 152 - 144 = 8)$$
$$18 = 2 \cdot 8 + 2, \quad 2 = \tilde{d} \qquad 8 = 4 \cdot 2 + 0$$

$$\tilde{d} = 2 = 18 - 2 \cdot 8 = 18 - 2 \cdot (512 - 28 \cdot 18)$$
$$= 57 \cdot 18 + (-2) \cdot 512$$
$$= 57(3 \cdot 198 + (-2) \cdot 288) + (-2) \cdot 512$$
$$= \underbrace{171}_{x} \cdot 198 + \underbrace{(-114)}_{y} \cdot 288 + \underbrace{(-2)}_{z} \cdot 512.$$

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§2.5: 6, 8b); §3.1: 4, 7; §3.2: 4, 13a) b); §3.3: 2, 26a) b)

6 §2.5 6):

Let x = # of calves; y = # of lambs; z = # of piglets.
Then
$$x+y+z=100; \quad 120x+50y+25z=4000.$$

Hence
$$120x+50y+25(100-x-y)=4000$$

$$95x+25y=1500 \quad \text{---} (*)$$

$$\Rightarrow \quad 19x+5y=300; \quad \gcd(19,5)=1$$

$$19=3\cdot 5+4 \qquad 5=4\cdot 1+1$$

$$\Rightarrow \quad 1=5-4\cdot 1 = 5-(19-3\cdot 5) = 4\cdot 5+(-1)\cdot 19.$$

Hence
$$5=4\cdot 25+(-1)\cdot 95$$
$$1500=1200\cdot 25+(-300)\cdot 95$$

Special solution to the Diophantine equations:
$$x=-300; \quad y=1200$$

Since $\gcd(95,25)=5$, the general solution to (*) is
$$x=-300+\frac{25}{5}t=-300+5t; \quad y=1200-\frac{95}{5}t=1200-19t.$$

$$x>0 \Rightarrow -300+5t>0; \quad t>\frac{300}{5}=60$$
$$y>0 \Rightarrow 1200-19t>0; \quad t<\frac{1200}{19}=63.2$$

Hence $t=61, 62, 63$. In these cases, we have

$(x,y,z)=(5,41,54); (10,22,68); (15,3,82)$.

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§2.5 8b) Let $x = \#$ of fruits in each pile
$y = \#$ of fruits each traveler gets.

Then
$$63x + 7 = 23y$$
$$\Rightarrow \quad -63x + 23y = 7 \quad \text{---} (*)$$

gcd(63, 23) = 1.

$$23 \overline{)63} \quad 46 \quad 17$$
$63 = 2 \cdot 23 + 17$

$$17 \overline{)23} \quad 17 \quad 6$$
$23 = 17 + 6$

$$6 \overline{)17} \quad 12 \quad 5$$
$17 = 2 \cdot 6 + 5$

$$5 \overline{)6} \quad 5 \quad 1$$
$6 = 5 \cdot 1 + 1$

$$\begin{aligned} 1 &= 6 - 5 \\ &= 6 - (17 - 2 \cdot 6) = -17 + 3 \cdot 6 \\ &= -17 + 3 \cdot (23 - 17) \\ &= 3 \cdot 23 - 4 \cdot 17 = 3 \cdot 23 - 4 \cdot (63 - 2 \cdot 23) \\ &= 11 \cdot 23 + (-4) \cdot 63 \end{aligned}$$

$$\Rightarrow \quad 7 = 77 \cdot 23 + (-28) \cdot 63$$

special solution to (*): $x_0 = 28$; $y_0 = 77$.

general solution $\quad x = 28 + 23t$; $\quad y = 77 + 63t$

$x > 0 \Rightarrow 28 + 23t > 0 \Rightarrow t > -\frac{28}{23} \approx -1.2$

$y > 0 \Rightarrow 77 + 63t > 0 \Rightarrow t > -\frac{77}{63} \approx -1.2$

Hence t can take any integer ≥ -1.

For example, when $t = -1$; $x = 5$; $y = 14$

$t = 0$; $x = 28$, $y = 77$.

(*) **§3.1 #4** Let $p \geq 5$ be a prime. Then p must be of the form $p = 1 + 6k$ or $p = 5 + 6k$ $(k \in \mathbb{Z}, k \geq 1)$.

(All the integers of forms $6k$, $6k+2$, $6k+3$, $6k+4$ are composite numbers.)

when $p = 1 + 6k$:

$$p^2 + 2 = (1+6k)^2 + 2 = 36k^2 + 12k + 3 = 3(12k^2 + 4k + 1)$$

— a composite.

(B5)

When $p = 5+6k$:

$$p^2+2 = (5+6k)^2+2 = 36k^2+60k+27 = 3(12k^2+20k+9)$$

— a composite.

§3.1 #7 Since $50! = 1\cdot 2\cdot 3\cdot \cdots \cdot 49\cdot 50$, all prime numbers that divide $50!$ are all primes ≤ 50. They are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

(4) §3.2 #4

4 a) Proving by contradiction, we assume $\sqrt{p}$ is a rational number. Then

$$\sqrt{p} = \frac{a}{b}, \quad a, b > 0, \quad \gcd(a,b)=1.$$

Then $p = \frac{a^2}{b^2} \Rightarrow p\cdot b^2 = a^2$.

Hence $p \mid a^2 \Rightarrow p \mid a$. Write $a = p\cdot c$.

Then $p\cdot b^2 = a^2 = p^2\cdot c^2 \Rightarrow b^2 = p\cdot c^2$.

Hence $p \mid b^2 \Rightarrow p \mid b$. Therefore $p \mid \gcd(a,b)$.

Hence $p=1$, contradicting the assumption that p is a prime.

3 b) Write $\sqrt[n]{a} = \frac{x}{y}$, $x, y > 0$, $\gcd(x,y)=1$.

Then $a = \frac{x^n}{y^n}$. $\Rightarrow x^n = a\cdot y^n$.

Hence $y \mid x^n \Rightarrow y \mid x\cdot x^{n-1}$. Since $\gcd(x,y)=1$,

$\Rightarrow y \mid x^{n-1}$. Proceeding in this way, we have
(Euclid's Lemma)

$y \mid x^0 \Rightarrow y=1$. Hence $\sqrt[n]{a} = x$, an integer. 6

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3 c) Proving by contradiction, we assume that $\sqrt[n]{n}$ is rational. By Part b), $\sqrt[n]{n}$ must be an integer.

Since $2^n > n$, we have $2 > \sqrt[n]{n}$. Hence $\sqrt[n]{n} = 1 \Rightarrow n = 1$. This contradicts the assumption that $n \geq 2$.

§3.2 #13

a) Since $n | m$, $m = nk$ for some $k > 0$. Hence

$$R_m = \frac{10^m - 1}{9} = \frac{1}{9}(10^{nk} - 1) = \frac{1}{9}((10^n)^k - 1)$$

$$= \frac{1}{9}(10^n - 1)\underbrace{((10^n)^{k-1} + (10^n)^{k-2} + \cdots + (10^n)^1 + 1)}_{a}$$

$$= R_n \cdot a$$

Hence $R_n | R_m$.

b) $$R_{n+m} = \frac{1}{9}(10^{m+n} - 1) = \frac{1}{9}(10^{m+n} - 10^m + 10^m - 1)$$

$$= \frac{1}{9}(10^{m+n} - 10^m) + \frac{1}{9}(10^m - 1)$$

$$= 10^m \cdot \frac{1}{9}(10^n - 1) + \frac{1}{9}(10^m - 1) = 10^m R_n + R_m.$$

Since $d | R_m$, $d | R_n$, we have $d | R_{n+m}$.

(8) §3.3 #2

4 (a) Let p, $p+2$ be twin primes. Then

$$1 + p(p+2) = p^2 + 2p + 1 = (p+1)^2$$ — a perfect square.

(B7) Since p is a prime,

(b) p must of the forms $p = 6k+1$ or $p = 6k+5$.

When $p = 6k+1$, Then $p+2 = 6k+1+2 = 6k+3 = 3(2k+1)$, — a composite number.

Therefore, p cannot be of form $6k+1$. Hence it must be of form $p = 6k+5$.

$$p + (p+2) = 2p + 2 = 12k + 10 + 2 = 12(k+1).$$

Hence the sum of twin primes p & $p+2$ is divisible by 12.

§3.3 #26

(a) All numbers ending with 33 are of the form:

$$33 + 100 \cdot k.$$

Since $\gcd(33, 100) = 1$, by the Dirichlet theorem, there are infinitely many of primes in this form.

(b) Consider positive integers of form

$$21k+5, \quad k = 1, 2, \ldots\ldots.$$

Since $\gcd(21, 5) = 1$, by the Dirichlet theorem, there are infinitely many primes in this form. We now prove that none of the twin primes have this form.

Assuming to the contrary, there is a twin prime $p = 21k+5$. Then $p+2 = 21k+5+2 = 7(3k+1)$ & $p-2 = 21k+3 = 3(7k+1)$ both are composite numbers. Hence p cannot be a twin prime! This contradiction proves the assertion!