(A1)

Solution to selected h.w. problems #1

§1.1: 10, 13; §1.2: 5; §2.2: 3, 8; §2.3: 14 b) c); 16, 20 a) b)

§1.1 #10 a) We prove the inequality by the mathematical induction principle (MIP).

When $n=1$, both the left and right hand sides equal to 1. Hence the inequality holds for $n=1$.

Assuming the inequality for $n=k$, we then have

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{k^2}\le 2-\frac{1}{k}.$$

Adding $\frac{1}{(k+1)^2}$ to both sides,

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}\le 2-\frac{1}{k}+\frac{1}{(k+1)^2}.$$

Since

$$(k+1)^2\ge(k+1)k,$$

$$\frac{1}{(k+1)^2}\le\frac{1}{(k+1)k}=\frac{1}{k}-\frac{1}{k+1}.$$

Therefore

$$-\frac{1}{k}+\frac{1}{(k+1)^2}\le-\frac{1}{k+1}.$$

Hence

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(k+1)^2}\le 2-\frac{1}{k+1}.$$

Therefore, the inequality holds for $n=k+1$.

By MIP, the inequality holds for all $n\ge1$.

b) When $n=1$, both the left hand and the right hand sides equal $\frac{1}{2}$. Hence the equality holds.

Assume the equality for $n=k$. Then

$$\frac{1}{2}+\frac{2}{2^2}+\cdots+\frac{k}{2^k}=2-\frac{k+2}{2^k}.$$

(A2)

Adding $\frac{k+1}{2^{k+1}}$ to both sides,

$$\frac{1}{2}+\frac{2}{2^2}+\cdots+\frac{k}{2^k}+\frac{k+1}{2^{k+1}} = 2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}$$

$$= 2-\frac{2(k+2)-k-1}{2^{k+1}}$$

$$= 2-\frac{k+3}{2^{k+1}}.$$

Hence the equality holds for $n=k+1$. By MIP, the equality holds for all $n \geq 1$.

§1.1 #13 We use the second mathematical induction principle.

When $n=1,2,3$, we have

$$a_1 = 1 < 2^1; \quad a_2 = 2 < 2^2; \quad a_3 = 3 < 2^3.$$

Hence the inequality holds for $n=1,2,3$.

Suppose the inequality holds for all $n \leq k$, $k \geq 4$, We now prove that it holds for $n=k+1$. By definition, we have:

$$a_{k+1} = a_k + a_{k-1} + a_{k-2}$$

$$< 2^k + 2^{k-1} + 2^{k-2}$$

$$= 2^{k+1}\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right)$$

$$= 2^{k+1}\cdot\frac{7}{8} < 2^{k+1}.$$

Hence the inequality holds for $n=k+1$. By MIP, it holds for all $n \geq 1$.

A3

§1.2 #5

(a) when $n=2$, $LHS = \binom{2}{2} = 1$, $RHS = \binom{3}{3} = 1$.

4 Hence the equality holds.

Assume now the equality for $n=k$. Then

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k}{2} = \binom{k+1}{3}.$$

Adding $\binom{k+1}{2}$ to both sides,

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k}{2} + \binom{k+1}{2} = \binom{k+1}{3} + \binom{k+1}{2}$$

$$= \binom{k+2}{3} \quad \left(\begin{array}{c}\text{Pascal's} \\ \text{identity}\end{array}\right)$$

Hence the equality holds for $n=k+1$. By MIP, it holds for all $n \geq 2$.

3 (b) Using $2\binom{m}{2} + m = \cancel{2} \cdot \frac{m(m-1)}{\cancel{2}} + m = m^2$,

we know from (a) that

$$1^2 + 2^2 + \cdots + n^2 = 2\left[\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2}\right] + (1+2+\cdots+n)$$

$$= 2\binom{n+1}{3} + \frac{n(n+1)}{2}$$

$$= \cancel{2} \cdot \frac{(n+1)n(n-1)}{3 \cdot \cancel{2}} + \frac{n(n+1)}{2}$$

$$= n(n+1)\left[\frac{n-1}{3} + 1\right] = \frac{n(n+1)(n+2)}{3}.$$

3 (c) Using $2\binom{m}{2} = 2 \cdot \frac{m(m-1)}{2} = m(m-1)$, we have

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = 2\left[\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n+1}{2}\right]$$

$$= 2 \cdot \binom{n+2}{3} = \cancel{2} \cdot \frac{(n+2)(n+1) \cdot n}{3 \cdot \cancel{2}} = \frac{n(n+1)(n+2)}{3}.$$

A4

§2.2 #3

(a) Any integer is of one of the following forms:

$n = 3q,\ 3q+1,\ 3q+2.$

when $n = 3q$, $\quad n^2 = (3q)^2 = 9q^2 = 3\cdot k; \quad k = 3q^2.$

$n = 3q+1, \quad n^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2+2q)+1$

$= 3k+1; \quad k = 3q^2+2q.$

$n = 3q+2, \quad n^2 = (3q+2)^2 = 9q^2 + 12q + 4$

$= 3(3q^2+4q+1) + 1 = 3k+1;$

$k = 3q^2+4q+1.$

Therefore in all cases, the square of an integer is either of form $3k$ or $3k+1$.

(b) As in part (a),

when $n = 3q$, $\quad n^3 = 27q^3 = 9k; \quad k = 3q^3.$

$n = 3q+1, \quad n^3 = 27q^3 + 3\cdot(3q)^2 + 3\cdot(3q) + 1$

$= 9(3q^3 + 3q^2 + q) + 1 = 9k+1, \quad k = 3q^3+3q^2+q.$

$n = 3q+2, \quad n^3 = 27q^3 + 3(3q)^2\cdot 2 + 3\cdot 3q\cdot 2^2 + 2^3$

$= 9(3q^3 + 6q^2 + 4q) + 8; \quad k = 3q^3 + 6q^2 + 4q.$

$= 9k+8$

Hence the cube of every integer is of one of the forms:

$9k,\ 9k+1,\ 9k+8.$

(c) Every integer is of one of the following forms:

$n = 5q,\ 5q+1,\ 5q+2,\ 5q+3,\ 5q+4.$

when $n = 5q$, $\quad n^4 = 5\cdot 5^3q^4 = 5\cdot 125q^4 = 5k; \quad k = 125q^4.$

(A5)

when

$n=5q+1$; $n^4 = (5q+1)^4 = (5q)^4 + 4(5q)^3 + 6(5q)^2 + 4(5q) + 1$

$= 5(\underbrace{5^3q^4 + 4\cdot5^2q^3 + 6\cdot5q^2 + 4q}_{k}) + 1$

$n=5q+2$: $n^4 = (5q+2)^4 = (5q)^4 + 4(5q)^3\cdot2 + 6(5q)^2\cdot2^2 + 4(5q)2^3 + \underset{16}{2^4}$

$= 5(\underbrace{5^3q^4 + 4\cdot5^2q^3\cdot2 + 6\cdot5\cdot q^2\cdot2^2 + 4\cdot q\cdot2^3 + 3}_{k}) + 1$

$n=5q+3$; $n^4 = (5q+3)^4 = (5q)^4 + 4(5q)^3\cdot3 + 6(5q)^2\cdot3^2 + 4(5q)\cdot3^3 + \underset{81}{3^4}$

$= 5(\underbrace{5^3q^4 + 4\cdot5^2q^3 + 6\cdot5q^2\cdot3^2 + 4\cdot q\cdot3^3 + 16}_{k}) + 1$

$n=5q+4$; $n^4 = (5q+4)^4 = (5q)^4 + 4(5q)^3\cdot4 + 6(5q)^2\cdot4^2 + 4(5q)\cdot4^3 + \underset{256}{4^4}$

$= 5(\underbrace{5^3q^4 + 4\cdot5^2\cdot q^3\cdot4 + 6\cdot5q^2\cdot4^2 + 4q\cdot4^3 + 51}_{k}) + 1$

Hence the 4^{th} power of an integer is either of form $5k$ or $5k+1$.

§2.2 #8 Every integer is either of form $2q$ or $2q+1$.

When $n=2q$, $n^2 = 4q^2 = 4k$, $k=q^2$

$n=2q+1$ $n^2 = 4q^2+4q+1 = 4(\underbrace{q^2+q}_{k})+1$.

Hence every perfect square is either of form $4k$ or $4k+1$.

(A6)

Now since

$$111\cdots1,111 = 111\cdots,108 + 3 = 4k+3,$$

it cannot be a perfect square.

(#) §2.3 #14 b)

Since

$$(-7)(5a+2) + 5(7a+3) = 1,$$

by Thm 2.4, $\gcd(5a+2, 7a+3) = 1$.

c) Let $d = \gcd(3a, 3a+2)$.

Since $-(3a) + (3a+2) = 2$, $d \mid 2$.

Hence either $d=2$ or $d=1$.

Since a is odd and $d \mid 3a$, d cannot be 2.

Hence $d=1$.

§2.3 #16

$$a^2 + (a+2)^2 + (a+4)^2 + 1$$

$$= a^2 + a^2 + 4a + 4 + a^2 + 8a + 16 + 1$$

$$= 3a^2 + 12a + 21 = 3(a^2 + 4a + 7)$$

Since a is odd, we write $a = 2q+1$, $q \in \mathbb{Z}$,

(A7)

Then

$$3(a^2+4a+7) = 3((2q+1)^2 + 4(2q+1)+7)$$
$$= 3[4q^2+4q+1+8q+4+7]$$
$$= 3[4q^2+12q+12] = 12(q^2+3q+3).$$

Hence $a^2+(a+2)^2+(a+4)^2+1$ is divisible by 12.

§2.3 #20 a)

Since $\gcd(a,b)=1$, $1 = ax+by$, $x, y \in \mathbb{Z}$

$\gcd(a,c)=1$, $1 = au+cv$, $u, v \in \mathbb{Z}$

Hence

$$1 = (ax+by)(au+cv)$$
$$= ax(au+cv) + by \cdot au + bcyv$$
$$= a\underbrace{[x(au+cv)+byu]}_{m} + bc\underbrace{yv}_{n}$$
$$= am + (bc)n.$$

By Thm 2.4, $\gcd(a, bc)=1$.

b) $\gcd(a,b)=1 \Rightarrow 1 = ax+by$, $x, y \in \mathbb{Z}$.

$c \mid a \Rightarrow a = cu.$

Hence $1 = c\underbrace{ux}_{m} + b\underbrace{y}_{n} = cm + bn$

By Thm 2.4, $\gcd(b,c)=1$.