b, a, d, b, c

Math 122 Test 2
Apirl 27, 2009

Your Name: Solutions

Part I. Multiple Choices: Circle the correct answer for each of the problem. Each problem in this part is worth 4 points.

(1) The arc length of $f(x) = \frac{2}{3}x^{3/2}$ over $[0, 1]$ is:

(a) 1/3 $(2\sqrt{2}-1)$

(b) 2/3 $(2\sqrt{2}-1)$

(c) 2/5 $(2\sqrt{2}-1)$

(d) 3/5. $(2\sqrt{2}-1)$

$$f'(x) = \frac{2}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} = x^{\frac{1}{2}}$$

$$\int_0^1 \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+x}\,dx \xlongequal[du=dx]{u=x+1} \int_1^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}}\Big|_1^2 = \frac{2}{3}(2\sqrt{2}-1)$$

(2) A dam is inclined at an angle of 30 degree. The height of the dam is 50 feet and the width of the base is 300 feet. The reservoir is filled with water. Assume that water density is w lb/ft^3. The water pressure on the slanted dam is:

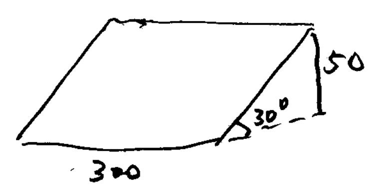
(a) 750000w lb

(b) 650000w lb

(c) 375000w lb

(d) 475000w lb

$f(y) = 300$



$$F = \int_0^{50} \frac{w}{\sin\theta}\, y\, f(y)\,dy$$

$$= \int_0^{50} \frac{w}{\frac{1}{2}}\cdot y\cdot 300\,dy = 600w\int_0^{50} y\,dy$$

$$= \cancel{600}\,w \times \cancel{\tfrac{1}{2}}\,50^2 = 750000\,w$$
$\quad 300$

(3) The center of mass of the lamina of constant density $\rho = 1$ g/cm^2 bounded by the region below $y = x^2$ over $[0, 1]$ is:

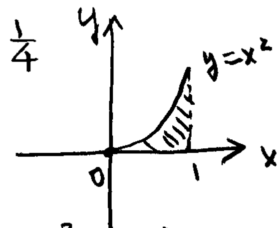
(a) (2/7, 4/5)

(b) (4/5, 2/7)

(c) (3/10, 3/4)

(d) (3/4, 3/10)

$$M_y = \rho \int_0^1 x\, f(x)\,dx = \int_0^1 x^3 dx = \frac{1}{4}$$



$$M_x = \rho \int_0^1 y(1-\sqrt{y})\,dy$$

$$= \int_0^1 (y - y^{\frac{3}{2}})\,dy = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

bigger space

(4) $\lim_{n\to\infty} \sqrt[n]{n} =$

(a) 0

(b) 1

(c) ∞

(d) e

$$M = \rho \int_0^1 f(x)\,dx = \int_0^1 x^2 dx = \frac{1}{3}.$$

$$x_{cm} = \frac{M_y}{m} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}; \quad y_{cm} = \frac{M_x}{m} = \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10}$$

$$y = x^{\frac{1}{x}} \qquad \ln y = \frac{\ln x}{x}, \quad \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1 \Rightarrow \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

(5) Find an N such that S_N approximates $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ with an error less than 10^{-4}.

(a) $N > 99$

(b) $N > 999$

(c) $N > 9999$

(d) $N > 99999$

$$|S_N - S| \le \frac{1}{N+1} < 10^{-4}$$

$$\Rightarrow N+1 > 10^4 \Rightarrow N > 10^4 - 1 = 9999$$

four

Part II. Solve ~~five~~ of the following six problems. You must show your work to get full credits. Each problem in this part is worth 10 points.

(1) Determine whether the improper integral converges. If so, evaluate it.

(a) $\displaystyle\int_0^\infty \frac{2\,dx}{\sqrt{x+2}}$ (b) $\displaystyle\int_0^1 x\ln x\,dx$

$$= \lim_{b\to\infty} \int_0^b \frac{2}{\sqrt{x+2}}\,dx \quad 1\text{ pt}$$

$$= \lim_{b\to\infty} 2\cdot 2(\sqrt{b+2}-\sqrt{2}) \quad 2\text{ pt}$$

$$= \infty \quad 2\text{ pt}$$

diverges.

$$\int x\ln x\,dx = \int \ln x\, d\left(\frac{x^2}{2}\right)$$

$$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4}. \quad 2\text{ pts}$$

$$\int_0^1 x\ln x\,dx = \lim_{a\to 0}\int_a^1 x\ln x\,dx$$

$$= \lim_{a\to 0}\left(-\frac{1}{4} - \left(\frac{a^2}{2}\ln a - \frac{a^2}{4}\right)\right)$$

$$= -\frac{1}{4}, \quad 1\text{ pts}$$ since $\lim_{a\to 0} a^2\ln a$

$$= \lim_{a\to 0}\frac{\ln a}{\frac{1}{a^2}} = \lim_{a\to 0}\frac{\frac{1}{a}}{-\frac{2}{a^3}} = \lim_{a\to 0}\left(-\frac{a^2}{2}\right) = 0. \quad 2\text{ pts}$$

(2) (a) Find the third degree Taylor polynomial $T_3(x)$ of $y = \ln x$ at $a = 1$.
(b) Find an n such that $|T_n(1.1) - \ln(1.1)| < 10^{-4}$.

(a) $f(x) = \ln x$, $f(1) = 0$

$f'(x) = x^{-1}$, $f'(1) = 1$

$f''(x) = -x^{-2}$, $f''(1) = -1$

$f'''(x) = 2x^{-3}$, $f'''(x) = 2$

2 pts

$$T_3(x) = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.$$

3 pts

(b) Since $f^{(n)}(x) = (-1)^{n+1}(n-1)!\,x^{-n}$, $|f^{(n+1)}(x)| \le n!$ on $[1, 1.1]$.

$$|T_n(x) - f(x)| \le \frac{K_{n+1}|x-a|^{n+1}}{(n+1)!} \quad 3\text{ pts}$$

$$= \frac{(0.1)^n}{n+1} < \frac{1}{10^4}. \quad 2\text{ pts}$$

Thus $n \ge 4$ would work.

(3) Find the sum of the following series:

(a) $\sum_{n=2}^{\infty} \frac{1}{n(n+1)}$ (b) $\sum_{n=0}^{\infty} \frac{3^n + 4^{n-2}}{5^n}$

$$S_N = \sum_{n=2}^{N} \frac{1}{n(n+1)}$$

$$= \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \quad 2pts$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= \frac{1}{2} - \frac{1}{N+1} \xrightarrow[2pt]{} \frac{1}{2}.$$

Thus $\sum_{n=2}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2}$. 1pt

$$= \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n + \frac{1}{4^2} \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$$

2pt

$$= \frac{1}{1-\frac{3}{5}} + \frac{1}{16} \cdot \frac{1}{1-\frac{4}{5}}$$

2pt

$$= \frac{5}{2} + \frac{1}{16} \cdot \frac{5}{1} = \frac{45}{16}$$

1pt

(4) Use the comparison or limiting comparison test to determine whether the given series is convergent.

(a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (b) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n}$

Let $a_n = \frac{\ln n}{n^3}$; $b_n = \frac{1}{n^2}$. 2pts

then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{\ln n}{n} \overset{x=n}{=} \lim_{x\to\infty} \frac{\ln x}{x}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{x} = 0. \quad 2pt$$

Since $\sum_{n=1}^{\infty} b_n$ converges $(p=2>1)$, by the limiting comparison Test, $\sum_{n=1}^{\infty} a_n$ also converges. 1pt

Since $\frac{2+(-1)^n}{n} \geq \frac{2-1}{n} = \frac{1}{n}$ 3pt

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p=1$), the series diverges by comparison test. 2pt

(5) Use the ratio or root test to determine whether the given series is convergent:

$$(a) \sum_{n=1}^{\infty} \frac{n^n}{n!} \qquad (b) \sum_{n=1}^{\infty} \frac{e^n}{n^n}$$

Let $a_n = \dfrac{n^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \quad 2pt$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n \to e > 1. \quad 2pt$$

By the ratio test, the series diverges. 1pt

Let $a_n = \dfrac{e^n}{n^n}$. Then

$$\sqrt[n]{a_n} = \frac{e}{n} \to 0 < 1. \quad 4pt$$

Therefore, by root test, the series converges. 1pt

(6) Find the interval of convergence for the power series.

$$(a) \sum_{n=1}^{\infty} \frac{2^n(x+1)^n}{n} \qquad (b) \sum_{n=1}^{\infty} n!\, x^n$$

(a) Let $b_n = \dfrac{2^n(x+1)^n}{n}$.

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{2^{n+1}|x+1|^{n+1}}{n+1} \cdot \frac{n}{2^n|x+1|^n}$$

$$= \frac{n}{n+1} \cdot 2|x+1| \to 2|x+1| < 1$$

$|x+1| < \frac{1}{2}$. 3 pt

$-\frac{3}{2}$ -1 $-\frac{1}{2}$

For $x = -\frac{1}{2}$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, diverges $(p=1)$. 1pt

For $x = -\frac{3}{2}$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges, by Alternating Series test. 1pt

Thus Interval of Convergence $I = [-\frac{3}{2}, -\frac{1}{2})$. 1pt

(b) Let $b_n = n!\, x^n$

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{(n+1)!\, |x|^{n+1}}{n!\, |x|^n}$$

$$= (n+1)|x| \to \infty > 1$$

except when $x = 0$. 3pt

Therefore, the interval of convergence is $I = \{0\}$, the center of the power series. 2pt