d, b, b, a, a

Math 122 Test 1
March 9, 2009

Your Name: Solutions

Part I. Multiple Choices: Circle the correct answer for each of the problem. Each problem in this part is worth 4 points.

(1) Let $f(x) = x^5 + x + 1$ and let g be the inverse of f. Then $g'(3) =$

(a) 1/3

(b) 1/4

(c) 1/5

(d) 1/6.

$f(x) = 3 \quad x^5 + x + 1 = 3$

$x^5 + x = 2 \quad x = 1 \quad g(3) = 1$

$$g'(3) = \frac{1}{f'(g(3))} = \left.\frac{1}{5x^4+1}\right|_{x=g(3)=1}$$

$$= \frac{1}{5 \cdot 1^4 + 1} = \frac{1}{6}$$

(2) $\lim_{x\to 0} \dfrac{\tan^{-1}(2x)}{\sin^{-1}(x)} = \left(\frac{0}{0}\right)$

(a) 1

(b) 2

(c) 1/2

(d) 1/3

$$= \lim_{x\to 0} \frac{\frac{1}{1+(2x)^2}\cdot 2}{\frac{1}{\sqrt{1-x^2}}} = \lim_{x\to 0} \frac{2\sqrt{1-x^2}}{1+(2x)^2} = \frac{2}{1} = 2$$

(3) The computer upgrade of a certain company will save \$10,000 a year for each of the next three year. Assume that the annual interest rate for loan is 6% and the savings is received as a lump sum at the end of each year. What is the present value of the savings?

(a) $10,000(e^{-0\times 0.06} + e^{-1\times 0.06} + e^{-2\times 0.06})$

(b) $10,000(e^{-1\times 0.06} + e^{-2\times 0.06} + e^{-3\times 0.06})$

(c) $10,000(e^{1\times 0.06} + e^{2\times 0.06} + e^{3\times 0.06})$

(d) $10,000(e^{0\times 0.06} + e^{1\times 0.06} + e^{2\times 0.06})$

(4) $\int_1^3 \ln x\, dx =$

(a) $3\ln 3 - 2$

(b) $3\ln 3 - 3$

(c) $2\ln 2 - 2$

(d) $2\ln 2 - 3$

$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x} dx$

$u = \ln x, \; dv = dx$

$du = \frac{1}{x} dx, \; v = x$

$= x \ln x - x \Big|_1^3 = 3\ln 3 - 3 - (1 \cdot \ln 1 - 1)$

$= 3\ln 3 - 3 - (-1)$

$= 3\ln 3 - 2$

(5) $\int \sin^3 x \cos^2 x\, dx =$

(a) $-\cos^3 x/3 + \cos^5 x/5 + C$

(b) $\cos^3 x/3 - \cos^5 x/5 + C$

(c) $-\sin^3 x/3 + \sin^5 x/5 + C$

(d) $\sin^3 x/3 - \sin^5 x/5 + C$

$\int \sin^2 x \cos^2 x \cdot \sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$= \int (1-u^2) u^2 \cdot (-du)$

$= -\int (u^2 - u^4)\, du = -\frac{u^3}{3} + \frac{u^5}{5} + C$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

(3) Find the limits.

(a) $\lim_{x\to 1}(1+\ln x)^{2/(x-1)}$,

$y = (1+\ln x)^{\frac{2}{x-1}}$

$\ln y = \dfrac{2}{x-1}\ln(1+\ln x)$

$\lim_{x\to 1} \dfrac{2\ln(1+\ln x)}{x-1}$ 2pt

$= \lim_{x\to 1} \dfrac{\frac{2}{1+\ln x}\cdot\left(+\frac{1}{x}\right)}{1}$ 2pt

$= +2$

$\lim_{x\to 1} y = e^2$ 1 pt

(b) $\lim_{x\to 0}\left(\csc x - \dfrac{1}{x}\right)$.

$= \lim_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$

$= \lim_{x\to 0} \dfrac{x-\sin x}{x\sin x}$ $\left(\frac{0}{0}\right)$ 2pt

$= \lim_{x\to 0} \dfrac{1-\cos x}{\sin x + x\cos x}$ $\left(\frac{0}{0}\right)$ 1 pt

$= \lim_{x\to 0} \dfrac{+\sin x}{\cos x + \cos x - x\sin x}$ 1pt

$= \dfrac{0}{2} = 0$ 1 pt

(4) The error bound of the mid-point rule is given by

$$\text{Error}(M_N) \le \frac{K_2(b-a)^3}{24N^2},$$

where K_2 is a constant such that $|f''(x)| \le K_2$. ~~Given~~

(a) Calculate M_4 for $\int_0^1 e^{x^2}\,dx$.

(b) Find a value of N for which $\text{Error}(M_N) \le 0.01$.

(You can leave your answer in terms of uncalculated numbers such as $\sqrt{2/3}, e^{2.5}$, etc.)

0.125 0.375 0.625 0.875

0 0.25 0.5 0.75 1

(a) $M_4 = \underset{\frac{1}{4}}{0.25}\left(e^{(0.125)^2} + e^{(0.375)^2} + e^{(0.625)^2} + e^{(0.875)^2}\right)$ ($0.125 = \frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$) 5 pts

≈ 1.449

(b) $f(x) = e^{x^2}$ $\quad f'(x) = 2x e^{x^2}$

$f''(x) = 2e^{x^2} + 4x^2 e^{x^2} = 2(1+2x^2)e^{x^2}$

$|f''(x)| \le 2(1+2)e^1 = \underset{K_2}{6e}$ on $[0,1]$ — 2 pts

$\dfrac{6e\cdot(1-0)^3}{24N^2} \le 0.01 \Rightarrow \dfrac{\cancel{6}e}{\cancel{24}_{4}}\cdot\cancel{100}^{25} \le N^2$

$N \ge 5\sqrt{e} \approx 8.24$. Take $N=9$. 3pts

Part II. Solve five of the following six problems. You must show your work to get full credits. Each problem in this part is worth 10 points.

(1) Use logarithmic differentiation to calculate the derivatives of

(a) $y = \cancel{x^{x}}$, x^{e^x}

$$\ln y = e^x \ln x \quad 2\text{pt}$$

$$\frac{y'}{y} = (e^x)\ln x + e^x \cdot \frac{1}{x} \quad 2\text{pt}$$

$$\Rightarrow y' = y\left(e^x \ln x + \frac{e^x}{x}\right)$$

$$= x^{e^x}\left(e^x \ln x + \frac{e^x}{x}\right)$$

$$= e^x x^{e^x}\left(\ln x + \frac{1}{x}\right) \quad 1\text{pt}$$

(b) $y = \sqrt{\dfrac{x(x+1)}{(2x+1)(-3x+2)}}$

$$\ln y = \frac{1}{2}\left[\ln x + \ln(x+1) - \ln(2x+1) - \ln(-3x+2)\right]$$

2 pts

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{1}{x} + \frac{1}{x+1} - \frac{2}{2x+1} + \frac{3}{-3x+2}\right]$$

2 pts

$$y' = \frac{1}{2}\sqrt{\frac{x(x+1)}{(2x+1)(-3x+2)}} \quad 1\text{pts}$$

$$\times\left[\frac{1}{x} + \frac{1}{x+1} - \frac{2}{2x+1} + \frac{3}{-3x+2}\right]$$

(2) The population of a city grows exponentially. Suppose that the doubling time is 10 years. How long does it take for the population to triple in size?

$$P(t) = P_0 e^{kt}; \quad \cancel{P_0} \cdot e^{k \cdot 10} = 2\cancel{P_0}$$

$$\Rightarrow k \cdot 10 = \ln 2 \Rightarrow k = \frac{\ln 2}{10}. \quad 3\text{pts}$$

$$P(t) = \cancel{P_0} e^{\frac{\ln 2}{10}t} = 3\cancel{P_0}$$

$$\frac{\ln 2}{10} t = \ln 3 \quad 2\text{pts}$$

$$t = \frac{\ln 3}{\ln 2} \cdot 10 \approx 15.85.$$

It takes about 15.85 years (15 years and 10 months) for the population to triple.

(5) Evaluate the integral:

$$\int \frac{dx}{x^2\sqrt{x^2-4}}.$$

$x = 2\sec\theta$ 3pts $x^2 - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta$

$$\text{Int} = \int \frac{\cancel{2}\sec\theta\,\cancel{\tan\theta}}{4\sec^2\theta\cdot\cancel{2}\,\cancel{\tan\theta}}\,d\theta = \frac{1}{4}\int\frac{1}{\sec\theta}\,d\theta$$

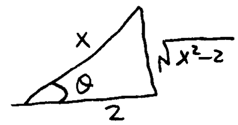
3pts

$$= \frac{1}{4}\int\cos\theta\,d\theta = \frac{1}{4}\sin\theta + C$$

2pts

$$= \frac{1}{4}\,\frac{\sqrt{x^2-2}}{x} + C$$

2pts



$\sec\theta = \frac{x}{2}$

(6) Evaluate the integral, using partial fractions:

$$\int \frac{5x^2 - x + 2}{(x-1)(x^2+1)}\,dx.$$

$$\frac{5x^2 - x + 2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

3pts

$$5x^2 - x + 2 = A(x^2+1) + (Bx+C)(x-1)$$
$$= Ax^2 + A + Bx^2 + (-B+C)x - C$$
$$= (A+B)x^2 + (-B+C) + A - C$$

$A + B = 5$ (1)   (1) + (2) : $A + C = 4$ – (4)

$-B + C = -1$ (2)   (4) + (3) : $2A = 6$

$A - C = 2$ (3)

$A = 3$

$B = 2$

$C = 1$ 3pts

$$\text{Int} = \int\frac{3}{x-1}\,dx + \int\frac{2x+1}{x^2+1}\,dx$$

2pts

$$= 3\ln|x-1| + \int\frac{2x}{x^2+1}\,dx + \int\frac{1}{x^2+1}\,dx$$

$$= 3\ln|x-1| + \ln(x^2+1) + \tan^{-1}(x) + C.$$

4pts