Math 122 Test 1-Take Home Makeup

March 9, 2009

Your Name:

Solve all seven problems. You must show your work to get full credits. Each problem is worth 10 points.

(1) Use logarithmic differentiation to calculate the derivatives of

$(a)\ y = x^{2^x}$, $\qquad$ $(b)\ y = \sqrt{\dfrac{x(x^2+1)}{(x^3+1)(x^4+1)}}$

(2) The population of a city grows exponentially. Suppose that the doubling time is 8 years. How long does it take for the population to triple in size?

(3) Find the limits.

$(a)\ \lim_{x\to 1}(1-\ln x)^{2/(x-1)}$, $\qquad$ $(b)\ \lim_{x\to 0}\left(\cot x - \dfrac{1}{x}\right)$.

(4) The error bound of the mid-point rule is given by

$$\text{Error}(M_N) \le \frac{K_2(b-a)^3}{24N^2},$$

where K_2 is a constant such that $|f''(x)| \le K_2$.

(a) Calculate M_4 for $\int_0^1 e^{x^4}\,dx$.

(b) Find a value of N for which $\text{Error}(M_N) \le 0.01$.

(You can leave your answer in terms of uncalculated numbers such as $\sqrt{2/3}, e^{2.5}$, etc.)

(5) Evaluate the integral:

$$\int \frac{dx}{x^2\sqrt{x^2-9}}.$$

(6) Evaluate the integral, using partial fractions:

$$\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx.$$

(7) Determine if each improper integral is convergent, and if so, evaluate it:

$(a)\ \int_0^1 x\ln x\,dx$, $\qquad$ $(b)\ \int_2^\infty \dfrac{dx}{(2x+2)^{2/3}}$.

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Math 122: Test 1 take-home Make up

(1) (a) $\ln y = 2^x \ln x$ 2 pts

$$\frac{y'}{y} = 2^x \ln 2 \cdot \ln x + 2^x \cdot \frac{1}{x} = 2^x \left(\ln 2 \ln x + \frac{1}{x} \right)$$

$$y' = y \cdot 2^x \left(\ln 2 \ln x + \frac{1}{x} \right) = x^{2^x} \cdot 2^x \left(\ln 2 \ln x + \frac{1}{x} \right)$$

3 pts

(b) $$\ln y = \frac{1}{2} \left[\ln x + \ln(x^2+1) - \ln(x^3+1) - \ln(x^4+1) \right]$$

2 pts

$$\frac{y'}{y} = \frac{1}{2} \left[\frac{1}{x} + \frac{2x}{x^2+1} - \frac{3x^2}{x^3+1} - \frac{4x^3}{x^4+1} \right]$$

$$y' = \frac{1}{2} \sqrt{\frac{x(x^2+1)}{(x^3+1)(x^4+1)}} \left[\frac{1}{x} + \frac{2x}{x^2+1} - \frac{3x^2}{x^3+1} - \frac{4x^3}{x^4+1} \right]$$

5 pts

(2) $P = P_0 e^{kt}$

$P(8) = \cancel{P_0} e^{8k} = 2 \cancel{P_0}$ $\qquad 8k = \ln 2; \qquad k = \frac{\ln 2}{8}$

4 pts

$$P = \cancel{P_0} e^{\left(\frac{\ln 2}{8}\right) t} = 3 \cancel{P_0}$$

$$\frac{\ln 2}{8} t = \ln 3 \implies t = \frac{\ln 3}{\ln 2} 8 \approx 12.6797$$

6 pts

(3)

(a) $$y = (1 - \ln x)^{\frac{2}{x-1}}$$

$$\ln y = \frac{2}{x-1} \ln(1 - \ln x) = \frac{2 \ln(1 - \ln x)}{x - 1} \quad \left(\frac{0}{0}\right)$$

2 pts

$$\lim_{x \to 1} \ln y \overset{L'H}{=} \lim_{x \to 1} \frac{2 \cdot \frac{1}{1 - \ln x} \cdot \left(-\frac{1}{x}\right)}{1} = \lim_{x \to 1} \frac{-2}{x(1 - \ln x)} = -2.$$

3 pts

12

Hence $\lim_{x\to 1} y = e^{-2}$

(b)

$$\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right)$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x} \quad \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 0} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{\sin x + x\cos x}$$

2pts

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0. \quad \left(\frac{0}{0}\right)$$

3pts

(4)

(a)

0.125 0.375 0.625 0.875

0 0.25 0.5 0.75 1

$$\int_0^1 e^{x^4}\,dx \approx \frac{1}{4}\left[e^{(0.125)^4} + e^{(0.375)^4} + e^{(0.625)^4} + e^{(0.875)^4}\right]$$

$$\approx 1.24654$$

5 pts

(b) $f'(x) = 4x^3 e^{x^4}$; $f''(x) = 12x^2 e^{x^4} + 16x^6 e^{x^4}$

$= (12x^2 + 16x^6) e^{x^4}$.

on $[0,1]$, $|f''(x)| \leq 28e = K_2$.

2pts

$$\frac{\overset{7}{\cancel{28}}\, e\,(1-0)^3}{\underset{6}{\cancel{24}}\, N^2} \leq 0.01$$

$$\Rightarrow N^2 \geq \frac{7}{6} e \times 100. \Rightarrow N \geq 10\sqrt{\frac{7e}{6}} \approx 17.8$$

3pts

Take $N \simeq 18$.

3

(5)

Let $x = 3\sec\theta$. Then

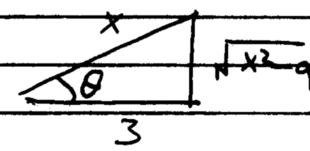
$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta \cdot 3\tan\theta}\,d\theta \quad \text{2pts}$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int \cos\theta\,d\theta$$

$$\sec\theta = \frac{x}{3}$$

$$= \frac{1}{9}\sin\theta + C \quad \text{2pts}$$

$$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C \quad \text{1 pt.}$$



(6)

$$\frac{3x^2-4x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$3x^2-4x+5 = A(x^2+1) + (Bx+C)(x-1)$$

$$= Ax^2 + A + Bx^2 + (-B+C)x - C$$

$$= (A+B)x^2 + (-B+C)x + A - C$$

$A + B = 3; \quad -①$

$-B + C = -4 \quad -②$

$A - C = 5 \quad -③$

①+②: $A + C = -1 \quad -④$

④+③: $2A = 4 \Rightarrow A = 2,$

From ①: $B = 1$; From ②: $C = -3$ 4 pts

$$\text{Hence } \int \frac{3x^2-4x+5}{(x-1)(x^2+5)}\,dx = \int \left(\frac{2}{x-1} + \frac{x-3}{x^2+1}\right)dx$$

$$= 2\int \frac{1}{x-1}\,dx + \int \frac{x}{x^2+1}\,dx - 3\int \frac{1}{x^2+1}\,dx$$

$$= 2\ln|x-1| + \frac{1}{2}\ln(x^2+1) - 3\arctan x + C \quad \text{6 pts}$$

(4)

(7)

(a) $\int x \ln x \, dx = \int \ln x \, d\left(\frac{x^2}{2}\right)$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2}\ln x - \frac{1}{4}x^2.$

2 pts

Hence

$\int_0^1 x \ln x \, dx = \lim_{a\to 0^+} \int_a^1 x \ln x \, dx$

$= \lim_{a \to 0^+} \left(-\frac{1}{4} - \left(\frac{a^2}{2}\ln a - \frac{1}{4}a^2 \right) \right)$

$= -\frac{1}{4} - \frac{1}{2} \lim_{a\to 0^+} a^2 \ln a.$

Since $\lim_{a\to0^+} a^2 \ln a = \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a^2}} \overset{L'H}{=} \lim_{a\to0^+} \frac{\frac{1}{a}}{-\frac{2}{a^3}} = \lim_{a\to0^+}\left(-\frac{a^2}{2}\right)$

$= 0,$

We know that $\int_0^1 x \ln x \, dx = -\frac{1}{4}$ — convergent.

3 pts

(b) $\int_2^\infty \frac{dx}{(2x+2)^{\frac{2}{3}}} = \lim_{b\to\infty} \int_2^b \frac{dx}{(2x+2)^{\frac{2}{3}}}$

$= \lim_{b\to\infty} \int_2^b (2x+2)^{-\frac{2}{3}} dx = \lim_{b\to\infty} 3(2x+2)^{\frac{1}{3}} \cdot \frac{1}{2} \Big|_2^b$

3 pt

$= \lim_{b\to\infty} \frac{3}{2}\left[(2b+2)^{\frac{1}{3}} - 8^{\frac{1}{3}} \right] = \infty.$

2 pt

Hence the integral diverges.

another method:

Since $x \ge 2$, $2x+2 \le 3x$. Hence

3 pts

$(2x+2)^{\frac{2}{3}} \le (3x)^{\frac{2}{3}} \Rightarrow \frac{1}{(2x+2)^{\frac{2}{3}}} \ge \frac{1}{(3x)^{\frac{2}{3}}} = \frac{1}{3^{\frac{2}{3}} \cdot x^{\frac{2}{3}}}$

Since $\int_2^\infty \frac{1}{x^{\frac{2}{3}}} dx$ diverges, $\int_2^\infty \frac{1}{(2x+3)^{\frac{2}{3}}} dx$ diverges by Comparison Test

2 pts