D1

Solution H.W #4

§11.1: 30, 39; §11.2: 11, 27; §11.3: 25, 40; §11.4: 10, 15; §11.5: 25, 38

§11.1 #30:

$$\lim_{n\to\infty}(\sqrt{n+3}-\sqrt{n}) = \lim_{n\to\infty}\frac{(\sqrt{n+3}-\sqrt{n})\cdot(\sqrt{n+3}+\sqrt{n})}{\sqrt{n+3}+\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{(n+3)-n}{\sqrt{n+3}+\sqrt{n}} = \lim_{n\to\infty}\frac{3}{\sqrt{n+3}+\sqrt{n}} = 0$$

§11.1 #39: Since

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \text{and } x = \tfrac{1}{n} \to 0 \text{ as } n\to\infty,$$

$$\lim_{n\to\infty} n\sin\tfrac{1}{n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1.$$

§11.2 #11 Since

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n},$$

We have

$$S_N = \sum_{n=3}^{N}\frac{1}{n(n-1)} = \sum_{n=3}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$= \left(\frac{1}{2}-\cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right) + \cdots \cancel{+} \left(\cancel{\frac{1}{N-1}}-\frac{1}{N}\right)$$

$$= \frac{1}{2}-\frac{1}{N}.$$

Therefore $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(n-1)} = \lim_{N\to\infty} S_N = \frac{1}{2}.$

§11.2 #27

$$\sum_{n=0}^{\infty}\frac{9\cdot 3^n + 4^{n-2}}{5^n} = 9\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n + 4^{-2}\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n$$

$$= 9\cdot\frac{1}{1-\frac{3}{5}} + \frac{1}{16}\cdot\frac{1}{1-\frac{4}{5}} = 9\cdot\frac{5}{2} + \frac{1}{16}\cdot\frac{5}{1}$$

$$= \frac{365}{16}$$ [Note: The textbook has $\frac{43^n}{5^n}$. In this case, the series diverges]

D2

§11.3 #25 Since

$$0 \le \frac{\sin^2 k}{k^2} \le \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges ( $p = 2 > 1$ ), by Comparison Test, the series $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$ converges.

§11.3 #40

Let $a_n = \frac{\ln n}{n^2}$ and $b_n = \frac{1}{n^{3/2}}$.

Then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{2}}}$$
$$= \lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{2}}} \left(\frac{\infty}{\infty}\right) \stackrel{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Since $\sum_{n=1}^{\infty} b_n$ converges ( $p = \frac{3}{2} > 1$ ), by the Limiting Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges.

§11.4 #10 Let $a_n = \tan \frac{1}{n}$. Then $a_n$ decreasing and $\lim_{n\to\infty} a_n = \tan 0 = 0$. Therefore, by the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

On the other hand, let $b_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} \stackrel{x=\frac{1}{n}}{=} \lim_{x\to 0} \frac{\tan x}{x}$$
$$\stackrel{L'H}{=} \lim_{x\to 0} \frac{\sec^2 x}{1} = 1.$$

D3

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges $(p=1)$,

by the Limiting comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \tan\frac{1}{n}$

diverges. Thus the series $\sum_{n=1}^{\infty} (-1)^n \tan\frac{1}{n}$ converges

conditionally.

* §11.4 #25

Let $a_n = n e^{-n}$; $b_n = \left(\frac{1}{2}\right)^n$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n e^{-n}}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty} \frac{n}{\left(\frac{e}{2}\right)^n}$$

$$\stackrel{x=n}{=\!=} \lim_{x\to\infty} \frac{x}{\left(\frac{e}{2}\right)^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{1}{\left(\frac{e}{2}\right)^x \ln\frac{e}{2}} = 0.$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges $(r = \frac{1}{2} < 1)$,

by the limiting comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n e^{-n}$

converges. Hence $\sum_{n=1}^{\infty} (-1)^n n e^{-n}$ converges

absolutely, and therefore it converges.

§11.4 #15 Let $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^4}$ and $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$.

Then $|S_N - S| < \frac{1}{(N+1)^4} < 0.0005 = 5 \times 10^{-4}$

$$\Rightarrow (N+1)^4 > \frac{1}{5} \times 10^4 \Rightarrow N > \frac{10}{\sqrt[4]{5}} - 1 \approx 5.7.$$

Take $N=6$, then

$$S_6 = +1 - \frac{1}{2^4} + \cdots - \frac{1}{6^4} \approx 0.947.$$

D4

**§11.5 #25** Let $a_n = \frac{n!}{n^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)\cdot n^n}{(n+1)^{n+1}}$$

$$= \frac{n^n}{(n+1)^n} = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n},$$

~~Hence~~ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$

By the ratio Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

**§11.5 #28**

Since $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0,$

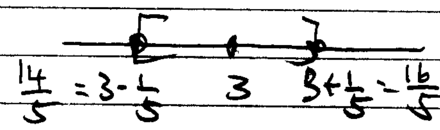
by the divergence test, the series diverges.

* **§11.6 #10** Let $b_n = \frac{(-5)^n (x-3)^n}{n^2}$. ~~Then~~

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \frac{5^{n+1}|x-3|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{5^n |x-3|^n}$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot 5|x-3| = 5|x-3| < 1.$$

~~Then $|x-3| < \frac{1}{5}$.~~




The power series converges absolutely on $\left(\frac{14}{5}, \frac{16}{5}\right)$.

When $x = \frac{14}{5}$, $\sum_{n=1}^{\infty} \frac{(-5)^n (x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

When $x = \frac{16}{5}$, $\sum_{n=1}^{\infty} \frac{(-5)^n (x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Both series converges absolutely ($p = 2 > 1$). Hence interval of convergence $= \left[\frac{14}{5}, \frac{16}{5}\right]$.