(C1) H.W. Solutions §9.1 – §9.4

§9.1: 17, 35; §9.2: [illegible], [illegible]; §9.3: 5; §9.4: 24, 30, 37

§9.1 #17

$$y^2 = \frac{1}{9}x(x-3)^2; \quad y \ge 0, \quad 0 \le x \le 3$$

$$\Rightarrow y = \frac{1}{3}\sqrt{x}\cdot|x-3| = \frac{1}{3}\sqrt{x}(3-x) = \sqrt{x} - \frac{x^{\frac{3}{2}}}{3}$$

$$y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} = \frac{1}{2}(x^{-\frac{1}{2}} - x^{\frac{1}{2}}).$$

$$(y')^2 = \frac{1}{4}(x^{-1} - 2 + x)$$

Hence

$$l = \int_0^3 \sqrt{1+(y')^2}\,dx = \int_0^3 \sqrt{1+\frac{1}{4}(x^{-1}-2+x)}\,dx$$

$$= \int_0^3 \sqrt{\frac{1}{4}(x^{-1}+2+x)}\,dx = \int_0^3 \sqrt{\frac{1}{4}(x^{-\frac{1}{2}}+x^{\frac{1}{2}})^2}\,dx$$

$$= \frac{1}{2}\int_0^3 (x^{-\frac{1}{2}}+x^{\frac{1}{2}})\,dx = \frac{1}{2}\left(2x^{\frac{1}{2}}\Big|_0^3 + \frac{2}{3}x^{\frac{3}{2}}\Big|_0^3\right)$$

$$= \frac{1}{2}\left(2\sqrt{3} + \frac{2}{3}3^{\frac{3}{2}}\right) = 2\sqrt{3}.$$

§9.1 #35

$$A = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi\int_0^8 (4-x^{\frac{2}{3}})^{\frac{3}{2}}$$

$$\times\sqrt{1+\left[\frac{3}{\cancel{2}}(4-x^{\frac{2}{3}})^{\frac{1}{2}}\cdot\left(-\frac{\cancel{2}}{3}x^{-\frac{1}{3}}\right)\right]^2}\,dx$$

$$= 2\pi\int_0^8 (4-x^{\frac{2}{3}})^{\frac{3}{2}}\cdot\sqrt{1+(4-x^{\frac{2}{3}})\cdot x^{-\frac{2}{3}}}\,dx$$

(C2)

$$= 2\pi \int_0^8 (4 - x^{\frac{2}{3}})^{\frac{3}{2}} \sqrt{4x^{-\frac{2}{3}}}\, dx$$

$$= 4\pi \int_0^8 (4 - x^{\frac{2}{3}})^{\frac{3}{2}} \cdot x^{-\frac{1}{3}}\, dx \qquad \left(y = 4 - x^{\frac{2}{3}}\right)$$

$$dy = -\tfrac{2}{3} x^{-\frac{1}{3}} dx$$

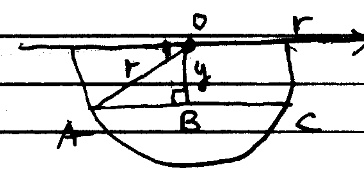
$$= 4\pi \int_4^0 y^{\frac{3}{2}} \cdot \left(-\tfrac{3}{2}\right) dy$$

$$= 6\pi \int_0^4 y^{\frac{3}{2}} dy = 6\pi \cdot \tfrac{2}{5}\, y^{\frac{5}{2}} \Big|_0^4$$

$$= \frac{384\pi}{5}$$

§9.2 #5

(a) By Pythagorean's thm,



$$\overline{AB} = \sqrt{r^2 - y^2}$$

Hence the width $\overline{AC} = 2\overline{AB} = 2\sqrt{r^2 - y^2}$.

(b)

$$F = \int_0^r \omega y f(y)\, dy = 2\omega \int_0^r y\sqrt{r^2 - y^2}\, dy$$

(ω = 9800 Newton/m³)

Let $u = r^2 - y^2$, $du = -2y\,dy$

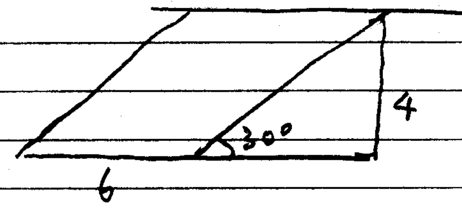
$$F = 2\omega \int_{r^2}^0 \sqrt{u} \cdot \left(-\tfrac{1}{2}\right) du = \omega \int_0^{r^2} u^{\frac{1}{2}} du = \omega\, \tfrac{2}{3} u^{\frac{3}{2}} \Big|_0^{r^2}$$

$$= \frac{2\omega}{3} r^3 \qquad \blacksquare$$

C3

§9.2 #18

$f(y) = 6, \quad \theta = \frac{\pi}{6}$



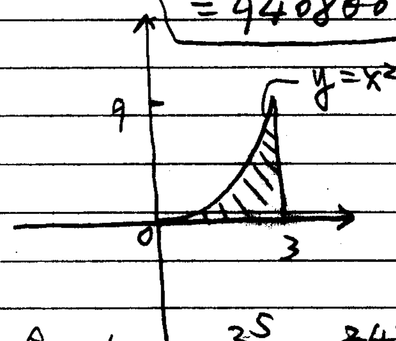
$$F = \int_0^4 \frac{w}{\sin\theta}\, y f(y)\, dy$$

$$= \frac{w}{\sin 30^\circ} \int_0^4 6y\, dy = 2w \left. 3y^2 \right|_0^4 = 96w = 96 \cdot 62.5$$

$$= \boxed{6000 \text{ lb}}$$

or $\boxed{96 \times 9800 = 940800 \text{ N}}$

§9.3 #5 $\rho = 3$ g/cm^2



(a) $$M_y = \rho \int_0^3 x(x^2 - 0)\, dx$$

$$= \rho \int_0^3 x^3 dx = \frac{\rho}{4} x^4 \Big|_0^3 = \frac{\rho}{4} \cdot 3^4 = \frac{3^5}{4} = \frac{243}{4}$$

— using (1)

$$M_x = \rho \int_0^9 y(3 - \sqrt{y})\, dy = \rho \int_0^9 (3y - y^{\frac{3}{2}})\, dy$$

$$= \rho \left(\frac{3}{2} y^2 - \frac{2}{5} y^{\frac{5}{2}} \right) \Big|_0^9 = \rho \left(\frac{3}{2} \cdot 81 - \frac{2 \cdot 3^5}{5} \right)$$

$$= 3 \cdot 81 \cdot \left(\frac{3}{2} - \frac{6}{5} \right) = \frac{3 \cdot 81 \cdot 3}{10} = \frac{729}{10}.$$

(b) $$m = \rho \int_0^3 f(x)\, dx = 3 \int_0^3 x^2 dx = 3 \cdot \frac{1}{3} x^3 \Big|_0^3 = 27$$

$$A = \int_0^3 f(x)\, dx = \int_0^3 x^2 dx = 9$$

$$x_{cm} = \frac{M_y}{m} = \frac{243/4}{27} = \frac{9}{4}; \quad y_{cm} = \frac{M_x}{m} = \frac{729}{10 \cdot 27} = \frac{27}{10}$$

Center of Mass $= \left(\frac{9}{4}, \frac{27}{10} \right)$.

(C4)

§9.4 #24

$f(x) = \sqrt{x};\qquad f(1)=1$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}\qquad f'(1) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\cdot(-\frac{1}{2})x^{-\frac{3}{2}};\qquad f''(1) = -\frac{1}{2^2}$

$f'''(x) = \frac{1}{2}(-\frac{1}{2})\cdot(-\frac{3}{2})x^{-\frac{5}{2}};\qquad f'''(1) = \frac{1\cdot 3}{2^3} = \frac{3!!}{2^3}$

$\cdots$

$f^{(n)}(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})x^{-\frac{(2n-1)}{2}}$

$= \frac{(-1)^{n-1}\cdot(2n-3)!!}{2^n}x^{-\frac{2n-1}{2}};\qquad f^{(n)}(1) = (-1)^{n-1}\cdot\frac{(2n-3)!!}{2^n}$

Hence $T_n(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{2^2}(x-1)^2 + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{2^n}(x-1)^n$

[Notation:
$(2n-3)!! = 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)$]

#30 $f(x) = x^{11/2}\qquad f(1)=1$

$f'(x) = \frac{11}{2}x^{\frac{9}{2}}\qquad f'(1) = \frac{11}{2}$

$f''(x) = \frac{99}{4}x^{\frac{7}{2}}\qquad f''(1) = \frac{99}{4}$

$f'''(x) = \frac{693}{8}x^{\frac{5}{2}}\qquad f^{(3)}(1) = \frac{693}{8}$

$f^{(4)}(x) = \frac{3465}{16}x^{\frac{3}{2}}\qquad f^{(4)}(1) = \frac{3465}{16}$

$f^{(5)}(x) = \frac{10395}{32}x^{\frac{1}{2}}.$

C5

Hence

$$T_4(x) = 1 + \frac{11}{2}(x-1) + \frac{1}{2!}\cdot\frac{99}{4}(x-1)^2 + \frac{1}{3!}\cdot\frac{693}{8}(x-1)^3 + \frac{1}{4!}\cdot\frac{3465}{16}(x-1)^4.$$

$$= 1 + \frac{11}{2}(x-1) + \frac{99}{8}(x-1)^2 + \frac{231}{16}(x-1)^3 + \frac{1155}{128}(x-1)^4$$

Since $$|T_4(x) - f(x)| \le \frac{M_5\,|x-a|^5}{5!}$$

where $$|f^{(5)}(x)| = \frac{10395}{32}|x|^{\frac{1}{2}} \le \frac{10395}{32}\cdot(1.2)^{\frac{1}{2}} = M_5$$ on $[1, 1.2]$

Hence the error is

$$|T_4(x) - f(x)| \le \frac{10395}{32}\cdot(1.2)^{\frac{1}{2}}\cdot(1.2-1)^5\cdot\frac{1}{5!}$$

$$\approx 9.489 \times 10^{-4} \quad \blacksquare$$

§9.4 #37 Since

$$|T_n(x) - f(x)| \le \frac{M_{n+1}\cdot|x-a|^{n+1}}{(n+1)!}$$

where $|f^{(n+1)}(c)| \le M_{n+1}$ for all c between x & a.

In this case, $a=1$; $x=1.3$

$$f'(x) = x^{-1}; \quad f''(x) = (-1)x^{-2}; \quad \ldots, \quad f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$$

$$f^{(n+1)}(x) = (-1)^n\, n!\, x^{-(n+1)}$$

Hence on $[1, 1.3]$

$$|f^{(n+1)}(x)| \le n! = M_{n+1}.$$

$$\frac{M_{n+1}|x-a|^{n+1}}{(n+1)!} = \frac{n!\,(1.3-1)^{n+1}}{(n+1)!} = \frac{0.3^{n+1}}{n+1} < 10^{-4}$$

Try: $n=5$: $\frac{0.3^{5+1}}{5+1} \approx 1.215 \times 10^{-4} > 10^{-4}$

$n=6$: $\frac{0.3^{6+1}}{6+1} \approx 3.124 \times 10^{-5} < 10^{-4}$. $\boxed{\text{Take } n=6}$