Math 122—Test I

March 9, 2006

Your Name: Solutions

Solve exactly eight of the following ten problems. If you solve more than ten problem, you must indicate which ones are to be graded. Each problem worths 10 points.

(1) Let $f(x) = x^2 - 4x - 5$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = 0$.

$f(x) = x^2 - 4x - 5 = 0 \qquad (x-5)(x+1) = 0$

$x = 5, \ \cancel{x = -1}$ (since $x \geq 2$)

$f'(x) = 2x - 4$ 5pts

$$\left.\frac{df^{-1}}{dx}\right|_{x=0} = \frac{1}{f'(5)} = \frac{1}{10-4} = \frac{1}{6} \quad \text{5pts}$$

(2) Find the derivatives of the following functions.

(a) $y = \dfrac{x\sqrt{x^4+1}}{(x+1)^{3/5}}$

(b) $y = x^{\cos x}, \quad x > 0$

(a) $\ln y = \ln x + \frac{1}{2}\ln(x^4+1) - \frac{3}{5}\ln(x+1)$ 2pts

$$\frac{y'}{y} = \frac{1}{x} + \frac{4x^3}{2(x^4+1)} - \frac{3}{5}\cdot\frac{1}{x+1}$$

$$y' = \frac{x\sqrt{x^4+1}}{(x+1)^{3/5}}\left[\frac{1}{x} + \frac{2x^3}{x^4+1} - \frac{3}{5(x+1)}\right]$$ 3pts

(b) $\ln y = \cos x \ln x$ 2pts

$$\frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}$$

$$y' = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$ 3pts

(3) Solve the initial value problem:

$$\frac{dy}{dt} = e^t \sin(3e^t - 6), \quad y(\ln 2) = 0$$

$y = \int e^t \sin(3e^t - 6)\, dt = -\frac{1}{3} \cos(3e^t - 6) + C$ 4 pt 3 pts

$y(\ln 2) = -\frac{1}{3}\cos(0) + C = 0 \Rightarrow C = \frac{1}{3}$ 3 pts

$\Rightarrow y = -\frac{1}{3}\cos(3e^t - 6) + \frac{1}{3}$

(4) Evaluate the integrals

(a) $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

$= \int \frac{dx}{\sqrt{1-(x-2)^2}}$ 3 pts

$= \arcsin(x-2) + C$ 2 pts

(b) $\int_1^2 \frac{\log_2 x}{x}\, dx = \frac{1}{\ln 2}\int_1^2 \frac{\ln x}{x}\, dx$ 1 pts

$(u = \ln x, \; du = \frac{dx}{x}$

$= \frac{1}{\ln 2}\int_0^{\ln 2} u\, du$ 2 pt

$= \frac{1}{2\ln 2}\cdot(\ln 2)^2 = \frac{\ln 2}{2}$ 2 pts

(5) The population of a certain bacteria culture grows exponentially. At the end of 4 hours there are 10,000 bacteria and at the end of 6 hours there are 50,000. How many bacteria were there initially.

$y = y_0 e^{kt}$

$50{,}000 = 10{,}000\, e^{2k}$ $\quad 5 = e^{2k}$

$\ln 5 = 2k \quad k = \frac{1}{2}\ln 5$ 5 pts

$10{,}000 = y_0 e^{4k} = y_0 e^{4\cdot\frac{1}{2}\ln 5}$

$= y_0 e^{2\ln 5} = y_0 e^{\ln 5^2}$

$= 25 y_0$

$\Rightarrow y_0 = \frac{10{,}000}{25} = 400$ 5 pts

(6) Evaluate the integrals

(a) $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

$u=\sqrt{t}$ $\quad du=\frac{dt}{2\sqrt{t}}$ $\quad t=1,\ u=1$; $t=4,\ u=2$

$=2\int_1^2 e^u\,du = 2\,e^u\Big|_1^2$ 3pts

$=2(e^2-e)$ 2pts

(b) $\int \frac{x\,dx}{8x^2+2}$

$u=8x^2+2,\ du=16x\,dx$

$=\frac{1}{16}\int\frac{du}{u}$ 3 pts

$=\frac{1}{16}\ln|u|+C$

$=\frac{1}{16}\ln(8x^2+2)+C$ 2pts

(7) Evaluate the trigonometric integrals.

(a) $\int \sin^2(2x)\cos^3(2x)\,dx$

$u=\sin(2x),\ du=2\cos(2x)dx$

$=\int u^2\cdot(1-u^2)\cdot\frac{1}{2}du$ 3pts

$=\frac{1}{2}\int(u^2-u^4)du$

$=\frac{1}{2}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+C$ 2pts

$=\frac{\sin^3(2x)}{6}-\frac{\sin^5(2x)}{10}+C$

(b) $\int \sin(3x)\cos(2x)\,dx$

$=\frac{1}{2}\int(\sin 5x+\sin x)dx$ 2pts

$=\frac{1}{2}\left(-\frac{\cos 5x}{5}-\cos x\right)$

$=-\frac{\cos(5x)}{10}-\frac{\cos x}{2}+C$ 3pts

(8) Evaluate the integrals using integration by parts and/or substitutions.

(a) $\int_1^2 2x\ln x\,dx$

$=\int_1^2 \ln x\,d(x^2)$

$=x^2\ln x-\int \underbrace{x^2\cdot\frac{1}{x}}_{x}dx$ 3pts

$=\left(x^2\ln x-\frac{x^2}{2}\right)\Big|_1^2$

$=(4\ln 2-2)-(0-\frac{1}{2})$

$=4\ln 2-\frac{3}{2}$ 2pts

(b) $\int e^{\sqrt{2x+3}}\,dx$

$u=\sqrt{2x+3},\ 2x+3=u^2$

$x=\frac{1}{2}u^2-\frac{3}{2}$

$dx=u\,du$

$=\int e^u u\,du=\int u\,de^u$ 3pts

$=ue^u-\int e^u du$

$=ue^u-e^u+C$

$=e^{\sqrt{2x+3}}(\sqrt{2x+3}-1)+C$ 2pts

(9) Evaluate the integral using long division and partial fraction decomposition.

$$\int \frac{y^3+3y^2+1}{y^3+y}\,dy$$

$$\begin{array}{r} 1 \\ y^3+y \overline{) \; y^3+3y^2+1} \\ -) \; \underline{y^3 \quad + y} \\ 3y^2 - y + 1 \end{array}$$

$$\frac{3y^2-y+1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \quad 3\text{pts}$$

$$3y^2-y+1 = A(y^2+1) + (By+C)y$$

$$= (A+B)y^2 + Cy + A$$

$$\Rightarrow \quad 3 = A+B, \Rightarrow B = 2$$

$$-1 = C$$

$$1 = A$$

3 pts 4 pts

$$\text{Int} = \int \left(1 + \frac{1}{y} + \frac{2y-1}{y^2+1}\right) dy = y + \ln|y| + \int \frac{2y}{y^2+1}\,dy - \int \frac{1}{y^2+1}\,d$$

$$= y + \ln|y| + \ln(y^2+1) - \tan^{-1} y +$$

(10) Evaluate the integrals using trigonometric substitutions.

$$\int \frac{dx}{x^2\sqrt{x^2+1}}$$

$$x = \tan\theta, \quad dx = \sec^2\theta\, d\theta \quad \sqrt{x^2+1} = \sec\theta$$

$$= \int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\tan^2\theta \cdot \cancel{\sec\theta}} = \int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta$$

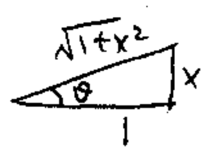
4 pts

$$= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta \quad \left(\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array}\right)$$

3 pts

$$= \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\sin\theta} = -\frac{\sqrt{1+x^2}}{x} + C$$

3 pts



Math 122—Test II

April 20, 2006

Your Name: Solutions

Solve exactly six of the following eight problems. If you solve more than six problem, you must indicate which ones are to be graded. Each problem worths 10 points.

(1) (a) Using the Simpson's Rule to estimate the integral $\int_1^2 \frac{1}{x^2}\,dx$ with $n = 4$ steps.
(b) How many steps are needed for the estimate to have error of magnitude less than 10^{-4}?

(a) 5pts $\Delta x = \frac{2-1}{4} = \frac{1}{4}$

x	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
y	1	$\frac{16}{25}$	$\frac{4}{9}$	$\frac{16}{49}$	$\frac{1}{4}$

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = \frac{1}{12}\left(1 + 4\cdot\frac{16}{25} + 2\cdot\frac{4}{9} + 4\cdot\frac{16}{49} + \frac{1}{4}\right) \quad 2\text{pts}$$

$$= 264821/529200 \approx 0.5 \quad 3\text{pts}$$

(b) $|E_S| \le \frac{M(b-a)^5}{180n^4}$ 2 pts $\quad M = \max\limits_{1\le x\le 2} |f^{(4)}|; \quad f(x) = x^{-2}$

$f'(x) = -2x^{-3}$ $\quad f'' = 6x^{-4}$ $\quad f''' = -24x^{-5}$ $\quad f^{(4)} = 120x^{-6}$; $\quad M = 120$: 2pts $\quad \frac{120\cdot 1^5}{180n^4} < 10^{-4}$ 1 pt

$\Rightarrow n^4 > \frac{120}{180}\cdot 10^4$

$n > 9.04$

$\boxed{n = 10}$

(2) Determine whether the following improper integrals are convergent. Show your work.

(a) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ 5 pts

method I:

$$\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{(1+x)\sqrt{x}}$$

$$\underset{du = \frac{dx}{2\sqrt{x}}}{\overset{u=\sqrt{x}}{=}} \lim_{b\to\infty} \int_1^{\sqrt{b}} \frac{2du}{1+u^2}$$

$$= \lim_{b\to\infty} 2\left(\tan^{-1}\sqrt{b} - \tan^{-1} 1\right)$$

$$= 2\cdot\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2} < +\infty$$

Converges

method II:

$$\frac{1}{(1+x)\sqrt{x}} \le \frac{1}{x\cdot\sqrt{x}} = \frac{1}{x^{3/2}}$$

$\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges (type I, $p = \frac{3}{2} > 1$)

$\Rightarrow \int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ converges by comparison

(b) $\int_0^\pi \frac{dx}{\sqrt{x} + \sin x}$ 5 pts

$0 \le \sin x \le 1$ for $0 \le x \le \pi$

$\Rightarrow \sqrt{x} + \sin x \ge \sqrt{x}$

$\Rightarrow 0 \le \frac{1}{\sqrt{x} + \sin x} \le \frac{1}{\sqrt{x}}$

$\int_0^\pi \frac{1}{\sqrt{x}}\,dx$ converges (type II, $p = \frac{1}{2} < 1$)

$\Rightarrow \int_0^\pi \frac{dx}{\sqrt{x} + \sin x}$ by comparison.

(3) Find the limits, if exist, of following sequences.

(a) $a_n = \left(\frac{n+1}{n-2}\right)^n$,

$\ln a_n = n \ln \frac{n+1}{n-2}$

$= n(\ln(n+1) - \ln(n-2))$

$\Rightarrow \lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln(n+1) - \ln(n-2)}{\frac{1}{n}}$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1} - \frac{1}{n-2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^2}{(n+1)(n-2)}$

$= \lim_{n\to\infty} \frac{3}{(1+\frac{1}{n})(1-\frac{2}{n})} = 3$

$\Rightarrow \lim_{n\to\infty} a_n = e^3$

(b) $a_n = \frac{(\ln n)^3}{n^2}$

$\lim_{n\to\infty} a_n \overset{L'H}{=} \lim_{n\to\infty} \frac{3(\ln n)^2 \cdot \frac{1}{n}}{2n}$

$= \lim_{n\to\infty} \frac{3(\ln n)^2}{2n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{6\ln n \cdot \frac{1}{n}}{4n}$

$= \lim_{n\to\infty} \frac{3\ln n}{2n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{3 \cdot \frac{1}{n}}{4n}$

$= \lim_{n\to\infty} \frac{3}{4n^2} = 0$

(4) Determine whether the following series converge. Find the sums of the convergent series.

(a) $\sum_{n=0}^{\infty} \left(\frac{1}{3^n} + \frac{(-1)^n}{2^n}\right)$

$= \sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$

$= (1 + \frac{1}{3} + \frac{1}{3^2} + \cdots) + (1 - \frac{1}{2} + \frac{1}{2^2} - \cdots)$

Geometric $a=1, r=\frac{1}{3}$, $a=1, r=-\frac{1}{2}$

$= \frac{1}{1-\frac{1}{3}} + \frac{1}{1+\frac{1}{2}}$

$= \frac{1}{\frac{2}{3}} + \frac{1}{\frac{3}{2}} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$

The series converges to $\frac{13}{6}$

(b) $\sum_{n=1}^{\infty} \frac{2}{(2n-1)(2n+1)}$.

$\frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}$ 2 pts

$S_n = (1 - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + \cdots + (\frac{1}{2n-1} - \frac{1}{2n+1})$

$= 1 - \frac{1}{2n+1}$

$S = \lim_{n\to} S_n = 1$

The Series converges to 1.

(5) Determine whether the series converges using the integral test.

$$\sum_{n=1}^{n} \frac{1}{n(1+(\ln n)^2)}.$$

$f(x) = \frac{1}{x(1+\ln^2 x)}$. Then $f(x) \geq 0$ & $f(x)$ is decreasing

2 pts

for $x \geq 1$.

$$\int_1^\infty \frac{dx}{x(1+\ln^2 x)} \xrightarrow[du = \frac{dx}{x}]{u = \ln x} \int_0^\infty \frac{du}{1+u^2} = \tan^{-1} u \Big|_0^\infty$$

$= \frac{\pi}{2} - 0 < +\infty \Rightarrow \sum_{n=1}^\infty \frac{1}{n(1+(\ln n)^2)}$ converges

by the integral Test.

(6) Determine whether the series converges using comparison tests.

(a) $\sum_{n=3}^{\infty} \frac{1+\sin^2 n}{n^2}$

$0 \leq \frac{1+\sin^2 n}{n^2} \leq \frac{1}{n^2}$

Since $\sum_{n=3}^\infty \frac{1}{n^2}$ converges

(p-series, $p = 2 > 1$),

$\sum_{n=3}^\infty \frac{1+\sin^2 n}{n^2}$ converges by the

Comparison Test.

(b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$.

$a_n = \frac{1}{\sqrt{n}\ln n}$, $b_n = \frac{1}{n}$

Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n}\ln n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$$

& $\sum b_n$ diverges (p-series, $p=1$), $\sum a_n$ also diverges by the limiting comparison Test.

(7) Determine whether the series converges using the ratio or root test.

(a) $\sum_{n=1}^{\infty} \frac{3^n}{n^2 \cdot 2^n}$

$$a_n^{\frac{1}{n}} = \frac{3}{(n^{\frac{1}{n}})^2 \cdot 2}$$

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{3}{2} > 1$$

$\Rightarrow \sum a_n$ diverges by the root Test.

(b) $\sum_{n=1}^{\infty} \frac{2^n n! n!}{(2n)!}$.

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{2^n n! \cdot n}$$

$$= \frac{2(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{2(1+\frac{1}{n})(1+\frac{1}{n})}{(2+\frac{1}{n})(2+\frac{2}{n})}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{2\cdot 2} = \frac{1}{4} < 1$$

$\Rightarrow \sum a_n$ converges by the ratio Test.

(8) Determine whether the series converges using a test of your choice.

(a) $\sum_{n=1}^{\infty} \frac{1}{n \sqrt[n]{n}}$

$$a_n = \frac{1}{n\sqrt[n]{n}}$$

$$b_n = \frac{1}{n}$$

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 < \infty$$

Since $\sum b_n$ diverges (p-series, $p=1$), $\sum a_n$ diverges as well.

(b) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

<u>method 1</u> $a_n = \ln(n+1) - \ln n$

$$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n)$$

$$= \ln(n+1) - \ln 1 = \ln(n+1)$$

(Telescope series)

$$\lim_{n\to\infty} S_n = \infty$$

The series diverges by definition.

<u>Method 2</u>: $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n$$

$= \ln e = 1$ & $\sum b_n$ diverges $\Rightarrow \sum a_n$ div b[y]