Math 122—Quiz I

February 2, 2006

Your Name: Solutions

Solve all four problems. Each problem worths 10 points.

(1) Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1$.

$f(x) = x^3 - 3x^2 - 1 = -1 \qquad x^3 - 3x^2 = 0 \qquad x^2(x-3) = 0$

$\Rightarrow \cancel{x=0},\ x = 3$ 3 pts

$$\left.\frac{df^{-1}}{dx}\right|_{-1} = \frac{1}{\left.\frac{df}{dx}\right|_3} = \left.\frac{1}{3x^2 - 6x}\right|_{x=3} = \frac{1}{9}$$

4 pts 3 pts

(2) Find the derivatives.

(a) $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

$\ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1)$ 2 pts

$\dfrac{y'}{y} = \dfrac{1}{x} + \dfrac{2x}{2(x^2+1)} - \dfrac{2}{3}\cdot\dfrac{1}{x+1}$

$y' = \dfrac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}\left(\dfrac{1}{x} + \dfrac{x}{x^2+1} - \dfrac{2}{3(x+1)}\right)$

3 pts

(b) $y = (\sin x)^x$

$\ln y = x \ln(\sin x)$ 2 pts

$\dfrac{y'}{y} = \ln(\sin x) + x\cdot\dfrac{\cos x}{\sin x}$

$y' = (\sin x)^x\left(\ln(\sin x) + x\cot x\right)$

3 pts

(3) Solve the initial value problem:

$$\frac{dy}{dt} = e^t \sin(e^t - 2), \quad y(\ln 2) = 0$$

$y = \int e^t \sin(e^t - 2)\,dt \quad (u = e^t - 2,\ du = e^t\,dt)$ 3 pts

$= \int \sin u\,du = -\cos u + C = -\cos(e^t - 2) + C$ 3 pts

$y(\ln 2) = -\cos(e^{\ln 2} - 2) + C = -\cos(2-2) + C$

$= -\cos 0 + C = -1 + C = 0 \Rightarrow C = 1$ 4 pts

$y = -\cos(e^t - 2) + 1$

(4) Evaluate the integrals

(a) $\int_0^{\pi/2} 7^{\cos t} \sin t\,dt$

$\left(\begin{array}{l} u = \cos t \\ du = -\sin t\,dt \\ t = 0,\ u = \cos 0 = 1 \\ t = \frac{\pi}{2} \quad u = \cos\frac{\pi}{2} = 0 \end{array}\right)$ 2 pts

$= -\int_1^0 7^u\,du = \int_0^1 7^u\,du$

$= \frac{1}{\ln 7} 7^u \Big|_{u=0}^{u=1} = \frac{1}{\ln 7}(7^1 - 7^0)$

$= \frac{6}{\ln 7}$ 3 pts

(b) $\int_1^4 \frac{\ln 2 \log_2 x}{x}\,dx$

$= \int_1^4 \frac{\cancel{\ln 2}\,\frac{\ln x}{\cancel{\ln 2}}}{x}\,dx$

$= \int_1^4 \frac{\ln x}{x}\,dx$ 3 pts

$u = \ln x$
$du = \frac{1}{x}dx$
$x = 1 \quad u = 0$
$x = 4 \quad u = \ln 4$

$= \int_0^{\ln 4} u\,du$

$= \frac{1}{2}u^2 \Big|_0^{\ln 4} = \frac{1}{2}(\ln 4)^2$

$= \frac{1}{2}(2\ln 2)^2 = 2(\ln 2)^2$

2 pts

1

Math 122 Quiz 2 Solutions

§7.5: 8, 21
§7.7: 70, 89
§8.1: 20, 39, 57
§8.2: 5, 16, 25

§7.5 #8 P_0 = initial population

$P(t)$ = Populations t hours later.

$P(3) = P_0 e^{r\cdot 3} = 10{,}000 \quad -①$

$P(5) = P_0 e^{r\cdot 5} = 40{,}000 \quad -②$

$\frac{②}{①}$: $\frac{P_0 e^{5r}}{P_0 e^{3r}} = 4 \Rightarrow e^{2r} = 4 \qquad 2r = \ln 4$

$\Rightarrow r = \ln 2.$

By ①: $P_0 e^{r\cdot 3} = 10{,}000$

$\Rightarrow P_0 = \frac{10000}{e^{3\ln 2}} = \frac{10{,}000}{2^3} = 1{,}250$

Ans: The initial population is 1,250

§7.5 #21 $H = H_s + (H_0 - H_s)e^{-kt}$

$H_s = 20; \quad H_0 = 90$

(a) $H(10) = 20 + 70e^{-k\cdot 10} = 60 \Rightarrow 70e^{-k\cdot 10} = 40$

$e^{-10k} = \frac{4}{7} \qquad k = -\frac{1}{10}\ln\frac{4}{7} = \frac{1}{10}(\ln 7 - \ln 4)$

$H(t) = 20 + 70e^{-k\cdot t} = 35$

$\Rightarrow e^{-kt} = \frac{15}{70} = \frac{3}{14} \qquad -kt = \ln\frac{3}{14}$

$t = +\frac{1}{k}(\ln 14 - \ln 3) = \frac{10(\ln 14 - \ln 3)}{\ln 7 - \ln 4} \approx 34.77 \text{ min}$

Ans: It takes about 35 minutes to cool down to 35°C

(b) $H_s = -15$

$H = -15 + 105e^{-k\cdot t} = 35$

$e^{-kt} = \frac{50}{105} = \frac{10}{21}$

$t = \frac{1}{k}(\ln 21 - \ln 10) = \frac{10(\ln 21 - \ln 10)}{\ln 7 - \ln 4} \approx 13.26 \text{ min}$

Ans: It takes about 13.26 minutes to cool down to 35°C in the freezer

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§7.7 #70

$$\frac{dy}{dx} = \frac{1}{x^2+4}\cdot 2x - \tan^{-1}\left(\frac{x}{2}\right) - x\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$$

$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2}$$

§7.7 #89

Let $u = \sin\theta$, $du = \cos\theta\, d\theta$ $\theta = -\frac{\pi}{2},\ u=-1$

$\theta = \frac{\pi}{2},\ u = 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos\theta\, d\theta}{1+(\sin\theta)^2} = 2\int_{-1}^{1}\frac{du}{1+u^2} = 2\arctan u\Big|_{-1}^{1} = 2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \pi$$

§8.1 #20

Let $u = \sqrt{t}$; $du = \frac{1}{2}t^{-\frac{1}{2}}dt = \frac{dt}{2\sqrt{t}}$

$$\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}} = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{t}} + C$$

§8.1 #39

$$\int \frac{dt}{\sqrt{-t^2+4t-3}} = \int\frac{dt}{\sqrt{1-(t-2)^2}} = \arcsin(t-2) + C$$

§8.1 #57

$$\int \frac{1}{1+\sin x}\,dx = \int\frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx = \int\frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$= \int\frac{1}{\cos^2 x}\,dx - \int\frac{\sin x}{\cos^2 x}\,dx$$

$$= \tan x + \int\frac{du}{u^2} \quad (u=\cos x)$$

$$= \tan x - \frac{1}{u} = \tan x - \frac{1}{\cos x} + C = \tan x - \sec x + C$$

3

§8.2 #5

$$\int_1^2 x\ln x\,dx = \int_1^2 \underbrace{\ln x}_{u}\, d\underbrace{\left(\frac{x^2}{2}\right)}_{v} = \frac{x^2}{2}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= 2\ln 2 - \frac{1}{2}\int_1^2 x\,dx = 2\ln 2 - \frac{1}{4}x^2\Big|_1^2 = 2\ln 2 - \frac{3}{4}$$

§8.2 #16

$$\int t^2 e^{4t}\,dt = \int \underbrace{t^2}_{u}\, d\underbrace{\left(\frac{1}{4}e^{4t}\right)}_{v} = \frac{1}{4}t^2e^{4t} - \frac{1}{2}\int e^{4t}\cdot t\,dt$$

$$= \frac{1}{4}t^2e^{4t} - \frac{1}{2}\int \underbrace{t}_{u}\, d\underbrace{\left(\frac{1}{4}e^{4t}\right)}_{v}$$

$$= \frac{1}{4}t^2e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{8}\underbrace{\int e^{4t}\,dt}_{=\frac{1}{4}e^{4t}}$$

$$= \frac{1}{4}e^{4t}\left(t^2 - \frac{1}{2}t + \frac{1}{8}\right) + C$$

§8.2 #25

Set $u = \sqrt{3s+9}$ $\quad u^2 = 3s+9$ $\quad s = \frac{1}{3}u^2 - 3$

$ds = \frac{2}{3}u\,du$

$$\int e^{\sqrt{3s+9}}\,ds = \int e^u\cdot\frac{2}{3}u\,du$$

$$= \frac{2}{3}\int u\,de^u = \frac{2}{3}\left(ue^u - \int e^u\,du\right)$$

$$= \frac{2}{3}\left(ue^u - e^u\right) + C = \frac{2}{3}\left(\sqrt{3s+9} - 1\right)e^{\sqrt{3s+9}} + C$$

HW1

H. W. #4

§8.8: 17, 25, 41, 51
§11.1: 65, ~~80~~, 81
§11.2: 34, 37, 71

§8.8 #17

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \underbrace{\int_0^1 \frac{dx}{(1+x)\sqrt{x}}}_{\text{type II}} + \underbrace{\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}}_{\text{type I}}$$

$$\int \frac{dx}{(1+x)\sqrt{x}} \; \frac{u=\sqrt{x}}{du=\frac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2\sqrt{x}}} \; 2\int \frac{du}{1+u^2} = 2\tan^{-1}u = 2\tan^{-1}(\sqrt{x})$$

Thus
$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{(1+x)\sqrt{x}} + \lim_{b\to\infty} \int_1^b \frac{dx}{(1+x)\sqrt{x}}$$

$$= \lim_{a\to 0^+} \big(2\underbrace{\tan^{-1}(1)}_{=\frac{\pi}{4}} - 2\underbrace{\tan^{-1}(\sqrt{a})}_{=0} \big)$$

$$+ \lim_{b\to\infty} \big(2\underbrace{\tan^{-1}(\sqrt{b})}_{=\frac{\pi}{2}} - 2\underbrace{\tan^{-1}(1)}_{=\frac{\pi}{4}} \big) = \pi$$

§8.8 #25

$$\int x \ln x \, dx \; \frac{\quad}{\substack{u=\ln x \\ v'=x \\ v=\frac{x^2}{2}}} \; \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{1}{4}x^2$$

$$\int_0^1 x\ln x\,dx = \lim_{a\to 0^+} \left(\frac{x^2}{2}\ln x - \frac{x^2}{4} \right)\Big|_a^1$$

$$= -\frac{1}{4} - \frac{1}{2}\lim_{a\to 0^+} x^2 \ln x = -\frac{1}{4}$$

$$\lim_{a\to 0^+} x^2\ln x = \lim_{a\to 0^+} \frac{\ln x}{\frac{1}{x^2}} \overset{L'H}{=} \lim_{a\to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{a\to 0} \left(-\frac{x^2}{2}\right) = 0$$

Ha2

§8.8 #41 Since $0 \le \sin t \le 1$ for $0 \le t \le \pi$, therefore

$$\sqrt{t} \le \sqrt{t} + \sin t$$

$$\frac{1}{\sqrt{t}+\sin t} \le \frac{1}{\sqrt{t}}$$

Since $\int_0^\pi \frac{1}{\sqrt{t}}\,dx$ is convergent (type II, $p = \frac{1}{2} < 1$)

by the comparison Test, $\int_0^\pi \frac{1}{\sqrt{t}+\sin t}\,dt$ is convergent

§8.8 #51 Take $g(x) = \frac{1}{\sqrt{x^6}} = \frac{1}{x^3}$

Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3/x^3}{\sqrt{x^6+1}/x^3}$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^6}}} = 1$$

$\int_0^\infty \frac{dx}{x^3}$ is convergent (Type I, $p = 3 > 1$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x^6+1}}$ is convergent (by Limiting comparison Test)

§11.1 #65 $\ln a_n = n \ln \frac{3n+1}{3n-1} = n(\ln(3n+1) - \ln(3n-1))$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln(3n+1) - \ln(3n-1)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{3}{3n+1} - \frac{3}{3n-1}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{-6}{9n^2-1}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{6n^2}{9n^2-1} = \lim_{n\to\infty} \frac{6}{9-\frac{1}{n^2}} = \frac{6}{9} = \frac{2}{3}$$

$\Rightarrow \lim_{n\to\infty} a_n = e^{\frac{2}{3}}$

HW3

§11.1 #80

$$\lim_{n\to\infty}\frac{(\ln n)^5}{\sqrt{n}} = \lim_{n\to\infty}\frac{5(\ln n)^4\cdot\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}} = \lim_{n\to\infty}\frac{10(\ln n)^4}{n^{\frac{1}{2}}}$$

$$= \lim_{n\to\infty}\frac{40(\ln n)^3\cdot\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}} = \lim_{n\to\infty}\frac{80(\ln n)^3}{n^{\frac{1}{2}}} = \lim_{n\to\infty}\frac{240(\ln n)^2\cdot\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}}$$

$$= \lim_{n\to\infty}\frac{480(\ln n)^2}{n^{\frac{1}{2}}} = \lim_{n\to\infty}\frac{960(\ln n)\cdot\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}}$$

$$= \lim_{n\to\infty}\frac{1920\ln n}{n^{\frac{1}{2}}} = \lim_{n\to\infty}\frac{1920\cdot\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}}$$

$$= \lim_{n\to\infty}\frac{3840}{\sqrt{n}} = 0$$

§11.1 #81

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(n-\sqrt{n^2-n}) = \lim_{n\to\infty}\frac{(n-\sqrt{n^2-n})(n+\sqrt{n^2-n})}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty}\frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}} = \lim_{n\to\infty}\frac{n}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty}\frac{1}{1+\sqrt{1-\frac{1}{n}}} = \frac{1}{1+\sqrt{1}} = \frac{1}{2}$$

§11.2 #34

$$a_n = \left(1-\frac{1}{n}\right)^n$$

$$\ln a_n = n\ln\left(1-\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\ln a_n = \lim_{n\to\infty}\frac{\ln\frac{n-1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{n-1}-\frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n\to\infty} -\frac{n^2}{(n-1)n}$$

$$= -1 \implies \lim_{n\to\infty} a_n = e^{-1} \neq 0$$

$\sum a_n$ is div by n^{th}-term Test.

HW4

§11.2 #37

$a_n = \ln\frac{n}{n+1} = \ln n - \ln(n+1)$

$S_1 = a_1 = \ln 1 - \ln 2 = -\ln 2$

$S_2 = a_1 + a_2 = (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \ln 3) = -\ln 3$

$\cdots$

$S_n = a_1 + a_2 + \cdots + a_n$

$= (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + \cdots + (\cancel{\ln n} - \ln(n+1))$

$= \ln 1 - \ln(n+1) = -\ln(n+1)$

$\lim_{n\to\infty} S_n = -\infty$ The series is divergent by definition.

§11.2 #71

$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$

$= \underbrace{(1 + r^2 + r^4 + r^6 + \cdots)}_{\substack{\text{Geometric} \\ \text{initial}=1,\ \text{ratio}=r^2}} + \underbrace{(2r + 2r^3 + 2r^5 + \cdots)}_{\substack{\text{Geometric} \\ \text{initial}=2r,\ \text{ratio}=r^2}}$

convergent when $|r|<1$

$= \frac{1}{1-r^2} + \frac{2r}{1-r^2} = \frac{1+2r}{1-r^2}$