

SEMI-CLASSICAL ANALYSIS OF SCHRÖDINGER OPERATORS AND COMPACTNESS IN THE $\bar{\partial}$ -NEUMANN PROBLEM

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ABSTRACT. We study the asymptotic behavior, in a “semi-classical limit”, of the first eigenvalues (i.e. the groundstate energies) of a class of Schrödinger operators with magnetic fields and the relationship of this behavior with compactness in the $\bar{\partial}$ -Neumann problem on Hartogs domains in \mathbb{C}^2 .

1. INTRODUCTION

Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . The complex Laplacian \square_q is the operator $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, acting as an (unbounded) self-adjoint operator on $\mathcal{L}_{0,q}^2(\Omega)$, the space of $(0, q)$ -forms with coefficients in $\mathcal{L}^2(\Omega)$. It is a classical result of Hörmander [H65] that \square_q has a bounded inverse. This inverse is the $\bar{\partial}$ -Neumann operator N_q . The $\bar{\partial}$ -Neumann operator is closely related to solving the $\bar{\partial}$ -equation and thus plays a central role in several complex variables. It is also of considerable interest from the point of view of partial differential equations, where it provides a prototype (of the solution) of an elliptic problem with non-coercive boundary conditions. For a detailed survey of the \mathcal{L}^2 -Sobolev regularity theory of the $\bar{\partial}$ -Neumann problem, we refer the reader to [BS99]. In particular, it is known that global regularity holds in many cases, but not all [Chr96]. A question closely related to global regularity is that of compactness of the $\bar{\partial}$ -Neumann operator. This question is of interest in its own right for a number of reasons; see [FS01] for a discussion of various aspects of the problem. In the context of global regularity, the relevance stems from a theorem of Kohn and Nirenberg [KN65] which implies that if N_q is compact in $\mathcal{L}_{(0,q)}^2(\Omega)$, then it is globally regular in the sense that it preserves the \mathcal{L}^2 -Sobolev spaces. Catlin [Ca84] demonstrated that compactness provides indeed a viable route to global regularity for the $\bar{\partial}$ -Neumann problem, the link being his concept of property (P). He showed that property (P) implies compactness (hence global regularity), and that it can be verified on large classes of domains. More recently, compactness is also being studied as a property not only stronger than global regularity, but one that is more robust and less subtle, and hence should be more amenable to a reasonable characterization in terms of properties of the boundary.

In this paper, we relate property (P) and compactness of the $\bar{\partial}$ -Neumann operator on complete pseudoconvex Hartogs domains in \mathbb{C}^2 to the asymptotic behavior of the groundstate energy of certain families of Schrödinger operators. It is well known

Date: December 27, 2001.

2000 Mathematics Subject Classification. 32W05, 35P15, 35J10, 35Q40.

The authors are supported in part by NSF grants DMS-0070697 and DMS-0100517 respectively.

that $\bar{\partial}$ - and related problems on such domains can be studied by means of the corresponding weighted problem on the base domain, see for example [L89], [Be94] and their references. In turn, studying the $\bar{\partial}$ -equation in weighted \mathcal{L}^2 -spaces on planar domains leads naturally to Schrödinger operators on these domains [Chr91],[Be96]. We show that compactness of the $\bar{\partial}$ -Neumann operator and property(P) on the Hartogs domain are characterized by the asymptotic behavior, in a “semi-classical limit”, of the lowest eigenvalues (the groundstate energies) of certain magnetic Schrödinger operators on the base domain and their non-magnetic counterparts, respectively.

To state the main result of this paper, we need to introduce some notation and recall some terminology. A compact set $K \subset \mathbb{C}^n$ is said to satisfy property (P) if for every positive number M , there exists a neighborhood U of K and a C^2 -smooth function λ on U , $0 \leq \lambda \leq 1$, such that for all z in K , the smallest eigenvalue of the Hermitian form $(\partial^2 \lambda(z) / \partial z_j \partial \bar{z}_k)_{j,k=1}^n$ is at least M . Let D be a bounded domain in \mathbb{C} and let $\phi \in C^2(\bar{D})$. Let $S_\phi = -[(\partial_x + i\phi_y)^2 + (\partial_y - i\phi_x)^2] + \Delta\phi$ be a magnetic Schrödinger operator and $S_\phi^0 = -\Delta + \Delta\phi$ be the corresponding non-magnetic Schrödinger operator. Let $\lambda_\phi(D)$ and $\lambda_\phi^0(D)$ be the first eigenvalues of the Dirichlet realization of S_ϕ and S_ϕ^0 on D respectively. (See Section 2 below for details.) We will also use the notation $\lambda(D)$ for $\lambda_{\phi=0}$, that is, for the lowest eigenvalue of minus the Dirichlet Laplacian.

Theorem 1. *Let $\Omega = \{(z, w) \in \mathbb{C}^2; z \in D, |w| < e^{-\phi(z)}\}$ be a smooth bounded complete pseudoconvex Hartogs domain in \mathbb{C}^2 . Suppose that $b\Omega$ is strictly pseudoconvex on $b\Omega \cap \{w = 0\}$. Then*

1. $b\Omega$ satisfies property (P) if and only if $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) = \infty$.
2. N is compact if and only if $\lim_{n \rightarrow \infty} \lambda_{n\phi}(D) = \infty$.

The necessity in Part 2 in Theorem 1 is implicit in Proposition 2 of Matheos’ paper [M97]. Note that $S_{n\phi} = -n^2[(\frac{1}{n}\partial_x + i\phi_y)^2 + (\frac{1}{n}\partial_y - i\phi_x)^2] + n\Delta\phi$. Letting n tend to infinity is thus analogous, in a sense, to letting “Planck’s constant” $h = 1/n$ tend to zero. Study of the latter situation is often referred to as semi-classical analysis (see e.g. [Hel88], chapter 1).

Global regularity is not an issue for the domains we study here: the $\bar{\partial}$ -Neumann problem is globally regular on any smooth bounded complete pseudoconvex Hartogs domain in \mathbb{C}^2 ([BS89]).

Sibony ([Si87], see also [Si91]) undertook a systematic study of property(P), under the name of B-regularity, on arbitrary compact sets in \mathbb{C}^n . Some of this work is also discussed in section 3 of [FS01]. In particular, in the situation of Theorem 1, $b\Omega$ satisfies property(P) (in \mathbb{C}^2) if and only if $W := \{z \in D | \Delta\phi = 0\}$ satisfies property(P) in the plane ([Si87], p.310, see also section 5 below).

We refer the reader to [FS01] for a detailed discussion of compactness in the $\bar{\partial}$ -Neumann problem. As mentioned above, on a bounded pseudoconvex domain, property(P) implies compactness of the $\bar{\partial}$ -Neumann operator. For sufficiently smooth domains, see [Ca84]; the second author observed in [St97] that no boundary regularity at all is needed. In light of Theorem 1, a quantitative way to look at this result

in the case of Hartogs domains in \mathbb{C}^2 is through (a special case of) Kato's inequality (see Proposition 3 below): $\lambda_{n\phi}(D) \geq \lambda_{n\phi}^0(D)$. It would be of considerable interest, both from the point of view of the $\bar{\partial}$ -Neumann problem and from that of Schrödinger operators, to determine whether or not conversely, the (equivalent) properties in part (2) of Theorem 1 imply those in part(1). For an example of a continuous (but nonsmooth) subharmonic ϕ where $\lim_{n \rightarrow \infty} \lambda_{n\phi}(D) = \infty$, but $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) < \infty$, see [ChrF01].

Recently, McNeal [McN01] showed that a variant of property(P) still implies compactness of the $\bar{\partial}$ -Neumann operator. For the domains we consider here, this variant turns out to be equivalent to property(P); we discuss this in the appendix (Section 5).

Acknowledgment. Part of this work was done while the first author visited Princeton University on an AMS Centennial Research Fellowship. The author is indebted to Professors J. J. Kohn and E. Lieb for stimulating discussions. He also thanks the Mathematics Department for hospitality.

2. SCHRÖDINGER OPERATORS

In this section, we collect some facts about the Schrödinger operators that arise in the setting of Theorem 1. The reader may in addition consult [Be96].

Let D be a bounded domain in \mathbb{C} and let $\phi(z) \in C^2(\bar{D})$. Let $\bar{L}_\phi = e^{-\phi} \frac{\partial}{\partial \bar{z}}(e^\phi \cdot) = \partial_{\bar{z}} + \phi_{\bar{z}}$ be the first order differential operator defined in the sense of distributions on $\mathcal{L}^2(D)$ and let $L_\phi = -e^\phi \frac{\partial}{\partial z}(e^{-\phi} \cdot) = -\partial_z + \phi_z$ be the (formal) adjoint of \bar{L}_ϕ . The domain of the actual adjoint of \bar{L}_ϕ is the Sobolev space $W_0^1(D)$. Note that \bar{L}_ϕ is just $\partial/\partial \bar{z}$ conjugated by multiplication by e^ϕ .

Consider the closed, positive semi-definite sesquilinear form

$$Q_\phi(u, v) = 4(L_\phi u, L_\phi v)$$

defined on $W_0^1(D) \times W_0^1(D) \subset \mathcal{L}^2(D) \times \mathcal{L}^2(D)$. Let S_ϕ be the unique non-negative, self-adjoint, densely defined operator on $\mathcal{L}^2(D)$ corresponding to $Q_\phi(u, v)$. For the connection between quadratic forms and (unbounded) self-adjoint operators, see for example [RS80], section VIII.6. Then $\text{Dom}(S_\phi) = W_0^1(D) \cap W^2(D)$, and on this domain

$$(1) \quad S_\phi = 4\bar{L}_\phi L_\phi = -[(\partial_x + i\phi_y)^2 + (\partial_y - i\phi_x)^2] + \Delta\phi.$$

This is the Dirichlet realization of the Schrödinger operator on D with magnetic potential $A = (-\phi_y, \phi_x) = -\phi_y dx + \phi_x dy$, magnetic field $dA = \Delta\phi dx \wedge dy$, and electric potential $V = \Delta\phi$. Since $\text{Dom}(S_\phi) = W_0^1(D) \cap W^2(D)$ embeds compactly into $\mathcal{L}^2(D)$, S_ϕ has compact resolvent. By construction, S_ϕ is the restriction to its domain of an isomorphism of $W_0^1(D)$ onto $W^{-1}(D)$. Consequently, as an (unbounded) operator on $\mathcal{L}^2(D)$, it is injective and onto, and so has a bounded inverse (which moreover is then compact). Let $S_\phi^0 = -\Delta + \Delta\phi$ be the Schrödinger operator without magnetic potential corresponding to S_ϕ (with the same domain as S_ϕ); it too has compact resolvent. Because both S_ϕ and S_ϕ^0 have compact resolvent, the spectrum in each

case consists of a sequence of eigenvalues tending to infinity. Let $\lambda_\phi(D)$ and $\lambda_\phi^0(D)$ be the first eigenvalues of S_ϕ and S_ϕ^0 , respectively. Note that

$$(2) \quad \begin{aligned} \lambda_\phi(D) &= \inf \left\{ 4 \int_D |L_\phi u|^2 dA / \int_D |u|^2 dA, \quad u \in C_0^\infty(D), u \not\equiv 0 \right\} \\ &= \inf \left\{ 4 \int_D |u_z|^2 e^{2\phi} dA / \int_D |u|^2 e^{2\phi} dA, \quad u \in C_0^\infty(D), u \not\equiv 0 \right\}. \end{aligned}$$

The case of most interest to us is that of a subharmonic ϕ (i.e. $\Delta\phi \geq 0$); this corresponds to pseudoconvexity of the Hartogs domain Ω . It turns out that subharmonicity of ϕ can be characterized in terms of the behavior of the lowest eigenvalues $\lambda_{t\phi}^0(D)$ and $\lambda_{t\phi}(D)$. The equivalence of (1) and (3) below is in [Be93].

Proposition 2. *The following are equivalent:*

1. $\Delta\phi \geq 0$;
2. $\liminf_{t \rightarrow \infty} \lambda_{t\phi}^0(D) > 0$;
3. $\limsup_{t \rightarrow \infty} \lambda_{t\phi}(D) > 0$.

Proof. (1) implies (2) because $S_{t\phi}^0 \geq -\Delta$ (as operators) when $\Delta\phi \geq 0$, and $-\Delta > 0$ (as operators). (2) implies (3) because $\lambda_{t\phi}(D) \geq \lambda_{t\phi}^0(D)$; for this, see Proposition 3 below. Finally, the proof that (3) implies (1) can be found in [Be93], proof of Proposition 1.5 and the remark immediately following that proof (bottom of page 212). \square

Remark 1. 1) Note that $\lambda(D)$ provides a lower bound on $\liminf_{t \rightarrow \infty} \lambda_{t\phi}^0(D)$ that is independent of ϕ , for ϕ 's that satisfy one of the properties in Proposition 2. In particular, the quantities in (2) and (3) in the proposition cannot be arbitrarily small positive for D given. $\lambda(D)$ itself can be estimated from below by $\pi/|D|$, where $|D|$ denotes the area of D ; this is a consequence of the Poincaré inequality (see e.g. [GT98], inequality (7.44)).

2) In general, one does not have $\limsup_{t \rightarrow \infty} \lambda_{t\phi}(D) = \liminf_{t \rightarrow \infty} \lambda_{t\phi}(D)$. For example, if D is an annulus and $\Delta\phi = 0$ on D then $\lambda_{t\phi}(D)$ is a periodic function of t with minimum $\lambda(D)$ and maximum $\lambda(D \setminus L)$ where L is a path connecting the components of bD (see [HHHO99], Theorem 1.1 and Remark 1.5 (vi)).

The first part of the following proposition is a special case of an inequality of Kato ([Ka72]; see also [Be96]). The second part goes back to [LO77]. A detailed proof of the proposition is in [Hel88], Lemma 7.2.1.2 and Theorem 7.2.1.1, to where we refer the reader.

Proposition 3. $\lambda_\phi(D) \geq \lambda_\phi^0(D)$. *Furthermore, equality holds if and only if (1) $\Delta\phi = 0$ on D and (2) $(1/2\pi) \int_\gamma A \in \mathbb{Z}$ for any simple closed smooth curve γ in D . In this case, $S_\phi(D)$ and $S_\phi^0(D)$ are unitarily equivalent via $u \leftrightarrow e^{ih}u$ for a $(\mathbb{R} \bmod 2\pi)$ -valued h with $dh = A$.*

The proof in [Hel88] uses the following identity from [LO77], which can be obtained by integration by parts:

$$(3) \quad 4\|L_\phi u\|^2 - \lambda_\phi^0(D)\|u\|^2 = \|(\partial_x + i\phi_y - u_x^0/u^0)u\|^2 + \|(\partial_y - i\phi_x - u_y^0/u^0)u\|^2.$$

Here, u_0 denotes the (normalized) eigenfunction of S_ϕ^0 to the eigenvalue λ_ϕ^0 . (This eigenvalue is known to be simple, and u_0 is known to be zero free in D , see e.g. [RS80], also the discussion in 7.2.1 in [Hel88].) In particular, the difference $\lambda_\phi(D) - \lambda_\phi^0(D)$ is the infimum of the right hand side of (3) over all $u \in C_0^\infty(D)$ with $\|u\| = 1$. Replacing ϕ by $n\phi$ in (3) gives a semi-explicit expression for $\lambda_{n\phi}(D) - \lambda_{n\phi}^0(D)$, which is the quantity of concern in the question of whether or not property(P) and compactness of the $\bar{\partial}$ -Neumann operator are actually equivalent for the domains in \mathbb{C}^2 considered in Theorem 1. We call the expression semi-explicit because it involves taking the infimum and it involves the eigenfunction u_0 , which depends on n . This latter dependence may possibly be mitigated by passing to subsequences which converge in appropriate senses; compare the discussion in Section 3. Nonetheless, formula (3) notwithstanding, it appears that determining whether or not $\lambda_{n\phi}(D)$ tending to infinity implies that $\lambda_{n\phi}^0(D)$ tends to infinity is a nontrivial matter. Note that if $W := \{z \in D \mid \Delta\phi(z) = 0\}$ has non-empty interior, both eigenvalues are bounded above, for each n , by the corresponding eigenvalues on, say, a disc contained in the interior of W . On such a disc, the magnetic and non-magnetic eigenvalues agree, by Proposition 3, and moreover, the non-magnetic eigenvalue is the same as that of $-\Delta$. So both sequences are bounded above (and thus fail to converge to infinity). Alternatively, if W contains a disc, $b\Omega$ contains an analytic disc, and both property(P) and compactness of the $\bar{\partial}$ -Neumann operator fail (see [FS01] for details). In contrast, when W has empty *fine* interior (see Section 3), then $b\Omega$ satisfies property(P) and the $\bar{\partial}$ -Neumann operator on Ω is compact so that both sequences of eigenvalues tend to infinity (in view of Theorem 1). Consequently, the case of interest is that of W with empty Euclidean interior, but non-empty fine interior. (When W has non-empty fine interior, there exists *some* smooth subharmonic function ψ , such that $W = \{z \in \mathbb{C} \mid \Delta\psi = 0\}$ and $\lim_{n \rightarrow \infty} \lambda_{n\psi}(D) < \infty$, see Remark 3 below.)

We point out that when no smoothness restrictions are placed on the boundary of Ω , compactness in the $\bar{\partial}$ -Neumann problem does not imply property(P). The domain $\{(z, w) \in \mathbb{C}^2 \mid 0 < |z| < 1, |z|^2 + |w|^2 < 1\}$, obtained by deleting from the unit ball the variety $\{z = 0\}$, has an analytic disc in its boundary (*a fortiori*, the boundary does not have property(P)), yet its $\bar{\partial}$ -Neumann operator is compact (see [FS01], example on page 150 preceding Proposition 4.1). The point is that the \mathcal{L}^2 -theory does not detect the deletion of the variety $\{z = 0\}$, and as a result, the $\bar{\partial}$ -Neumann operator inherits compactness from the $\bar{\partial}$ -Neumann operator on the unit ball. Recently, Christ and Fu ([ChrF01]) have constructed an example of a continuous subharmonic ϕ with $\nabla\phi \in \mathcal{L}^2(D)$, $\Delta\phi \in \mathcal{L}^1(D)$, and $\lim_{n \rightarrow \infty} \lambda_{n\phi}(D) = \infty$, but $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) < \infty$.

That magnetic Schrödinger operators majorize their non-magnetic counterparts in some appropriate sense, such as Kato's inequality, is generally referred to as diamagnetism, and the opposite direction (usually in terms of more general so called Pauli operators) is called paramagnetism. The property in question in the previous paragraph may thus be viewed as a paramagnetic property of the family of Schrödinger operators $\{S_{n\phi} \mid n \in \mathbb{N}\}$ and their non-magnetic counterparts $\{S_{n\phi}^0 \mid n \in \mathbb{N}\}$. It appears that what is known in the theory of Schrödinger operators in this direction concerns cases that, when specialized to the context of Theorem 1, cover situations that are

well understood from the point of view of the $\bar{\partial}$ -Neumann problem. For example, when the magnetic field is constant a result of Lieb (which is in terms of more general Pauli operators, see [AHS78]) implies that $\lambda_\phi(D) \leq \lambda_{2\phi}^0(D)$. Lieb's result was later generalized by Avron and Seiler to the case where the magnetic fields are given by certain polynomials ([AS79]). The ideas in the proofs of these results actually work when $\phi(z) = \sum_{j=1}^m |h_j(z)|^2$.

Proposition 4. *If $\phi = \sum_{j=1}^m |h_j(z)|^2$ where $h_j(z)$ are holomorphic on D , then*

$$(4) \quad \lambda_\phi(D) \leq \lambda_{2\phi}^0(D).$$

Proof. Let g be a real-valued eigenfunction of $S_{2\phi}^0$ corresponding to $\lambda_{2\phi}^0 = \lambda_{2\phi}^0(D)$. For $\zeta \in \mathbb{C}^m$, we let $H(z, \zeta) = -\sum_{j=1}^m (h_j(z)\zeta_j + |\zeta_j|^2)$, $\Psi(z, \zeta) = e^{-\phi+H(z, \zeta)}$, and $f = g\Psi$. It follows that $S_\phi(f) = \lambda_{2\phi}^0 f + 4(2\phi_z - H_z)\Psi g_{\bar{z}}$. Therefore,

$$\begin{aligned} (S_\phi(f), f) &= \lambda_{2\phi}^0 \|f\|^2 - 2 \int_D \frac{\partial |\Psi|^2}{\partial z} \frac{\partial g^2}{\partial \bar{z}} \\ &= \lambda_{2\phi}^0 \|f\|^2 + 2 \int_D \frac{\partial^2 |\Psi|^2}{\partial z \partial \bar{z}} g^2 \\ &= \lambda_{2\phi}^0 \|f\|^2 + 2 \int_D |f|^2 (|2\phi_z - H_z|^2 - 2\phi_{z\bar{z}}) \\ &= \lambda_{2\phi}^0 \|f\|^2 + 2 \sum_{j,k=1}^m \int_D h_{jz} \bar{h}_{kz} \frac{\partial^2 |\Psi|^2}{\partial \bar{\zeta}_j \partial \zeta_k} g^2. \end{aligned}$$

Denote $I(\zeta)$ the last term above. It follows from the divergence theorem that $\int_{\mathbb{C}^m} I(\zeta) = 0$. Therefore, there is a $\zeta_0 \in \mathbb{C}^m$ such that $I(\zeta_0) \leq 0$. We then conclude the proof. \square

Remark 2. Proposition 4 does not hold for all ϕ with $\Delta\phi \geq 0$. For example, if $D = \{z \in \mathbb{C}; 1/2 < |z| < 2\}$ and $\Delta\phi = 0$ on D , then $\lambda_{2\phi}^0(D) = \lambda(D) = \lambda_\phi^0(D)$, which is strictly less than $\lambda_\phi(D)$ unless $(1/2\pi) \int_{|z|=1} A \in \mathbb{Z}$ (Proposition 3). By a limiting process, one can in fact find examples such that D is the unit disc and $\Delta\phi \geq 0$ on D but (4) fails.

3. SOME POTENTIAL THEORY

Recall that the fine topology on \mathbb{C} is the weakest topology so that every subharmonic function is continuous. A general reference for the basic facts about the fine topology in \mathbb{C} is [He69]. We use int_f to denote the interior in the fine topology.

The Dirichlet problem for (minus) the Laplacian can be formulated on finely open sets; see [F99], section 3, and the references there for this formulation. The resulting theory inherits many features of the classical theory, but avoids some of its problems having to do with ‘‘stability’’ of sets (see [F99], Remark 2.4). What matters for us is the behavior of the first eigenvalue under a decreasing sequence of finely open sets. If U is finely open, we still use $\lambda(U)$ to denote this eigenvalue. Then, if $\{U_j\}_{j=0}^\infty$ is a decreasing sequence of bounded finely open sets in \mathbb{C} , $\{\lambda(U_j)\}_{j=0}^\infty$ is increasing (as

in the classical case), and (unlike the classical case) $\lim_{j \rightarrow \infty} \lambda(U_j) = \lambda(\text{int}_f(\cap_j U_j))$ ([F99], Theorem 2, part 1°).

The next proposition combines work of Fuglede and Sibony. We need it in the proof of part (1) of Theorem 1.

Proposition 5. *Let K be a compact subset of \mathbb{C} . The following are equivalent:*

1. K satisfies property (P).
2. K has empty fine interior.
3. K supports no non-zero function in $W_0^1(\mathbb{C})$.
4. For any open sets U_j such that $K \subset\subset U_{j+1} \subset\subset U_j$ and $\cap_{j=1}^{\infty} \overline{U_j} = K$, $\lambda(U_j) \rightarrow \infty$ as $j \rightarrow \infty$.

Proof. The equivalence of (1) and (2) is [Si87], Proposition 1.11. Let $\{U_j\}_{j=0}^{\infty}$ be a sequence of (Euclidean) open sets with $K \subset\subset U_{j+1} \subset\subset U_j$ and $\cap_j \overline{U_j} = K$. If (2) holds, then $\{0\} = W_0^1(\text{int}_f K) = \cap_{j=1}^{\infty} W_0^1(U_j)$ ([F99], Lemma 1.1, (ii)), which gives (3). If (3) holds, (4) must hold. If not, there would exist a sequence of functions $\{u_j\}_{j=1}^{\infty}$, $u_j \in W_0^1(U_j)$, with $\|u_j\| = 1$ and $\|\nabla u_j\| \leq \text{const.}$ (use (2) for $\phi \equiv 0$), for a suitable sequence $\{U_j\}_{j=1}^{\infty}$. Passing to a subsequence that converges both weakly in $W_0^1(\mathbb{C})$ and in norm in \mathcal{L}^2 of a neighborhood of K would yield a non-zero element of $W_0^1(\mathbb{C})$ that is supported on K , contradicting (3). Finally, (4) implies (2) because $\lim_{j \rightarrow \infty} \lambda(U_j) = \lambda(\text{int}_f(\cap_j U_j)) = \lambda(\text{int}_f K)$, and the last quantity is finite if $\text{int}_f K \neq \emptyset$, see [F99], Theorem 2, part 1°.

□

Remark 3. There are also characterizations of sets with empty fine interior in terms of logarithmic capacity ([He69], chapter 10, section 5) and in terms of Brownian motion ([Do84], section 2.IX.15). Our work shows that such a characterization can also be given in terms of non-magnetic Schrödinger operators: a compact set $K \subseteq \mathbb{C}$ has empty fine interior if and only if for every smooth subharmonic function ϕ on a domain D with $K \subset\subset D$, such that $K \supseteq \{z \in \mathbb{C} | \Delta \phi = 0\}$, $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) = \infty$. If K has empty fine interior, then combining Proposition 5 and (the proof of) part (1) of Theorem 1 shows that $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) = \infty$ for the ϕ 's under consideration. Conversely, the authors have shown ([FS01], Theorem 4.2) that if K has non-empty fine interior, there exists a smooth, bounded, pseudoconvex, complete Hartogs domain in \mathbb{C}^2 whose weakly pseudoconvex boundary points project onto K and whose $\bar{\partial}$ -Neumann operator is not compact. Moreover, the Hartogs domain can be chosen to satisfy the assumptions in Theorem 1. Consequently, the resulting function ϕ on the base domain D satisfies $\lim_{n \rightarrow \infty} \lambda_{n\phi}(D) < \infty$ (Theorem 1). *A fortiori*, $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) < \infty$ (Proposition 3).

4. PROOF OF THEOREM 1

Let $W = \{z \in D | \Delta \phi(z) = 0\}$. Because $b\Omega$ is strictly pseudoconvex near the points of the boundary where $w = 0$, W is a compact subset of D . To prove part (1) of Theorem 1, we use that $b\Omega$ satisfies property (P) if and only if W satisfies property (P) as a set in \mathbb{C} ([Si87], p. 310, see also section 4 below). In turn, W satisfies property (P)

if and only it satisfies (4) in Proposition 5. Fix a sequence $\{W_j\}_{j=1}^\infty$ of open subsets of D such that $W = \bigcap_j \overline{W_j}$, and $W \subset\subset W_{j+1} \subset\subset W_j \subset\subset D$.

Assume W satisfies (4) in Proposition 5. Let $\eta_j \in C_0^\infty(W_j)$, $0 \leq \eta_j \leq 1$, and $\eta_j = 1$ on W_{j+1} . For any $u \in C_0^\infty(D)$, $j \in \mathbb{N}$,

$$\begin{aligned} (S_{n\phi}^0 u, u) &= \|\nabla u\|^2 + n\|\sqrt{\Delta\phi}u\|^2 \\ &\geq \frac{1}{2}\|\nabla(u\eta_j)\|^2 + \frac{n}{2}\|\sqrt{\Delta\phi}u\|^2 \\ &\geq \frac{1}{2}\lambda(W_j)\|u\eta_j\|^2 + \frac{n}{2}\|\sqrt{\Delta\phi}u\|^2 \\ &\geq \frac{1}{2}\lambda(W_j)\|u\|^2 \end{aligned}$$

when n is sufficiently large. By assumption, $\lambda(W_j) \rightarrow \infty$ as $j \rightarrow \infty$. Therefore $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) = \infty$. This finishes one direction in the proof of part (1) of Theorem 1.

For the other direction, observe that for all $(n, j) \in \mathbb{N} \times \mathbb{N}$, $\lambda_{n\phi}^0(D) \leq \lambda_{n\phi}^0(W_j)$ (by the monotonicity with respect to the domain of the eigenvalue). Also, for $u \in C_0^\infty(W_j)$,

$$(S_{n\phi}^0 u, u) = (-\Delta u + n\Delta\phi u, u) \leq (-\Delta u, u) + (u, u)$$

if j is big enough relative to n so that $|n\Delta\phi| \leq 1$ on W_j . Consequently, $\lambda(W_j) \geq \lambda_{n\phi}^0(W_j) - 1 \geq \lambda_{n\phi}^0(D) - 1$ if j is big enough relative to n . It follows that $\lim_{j \rightarrow \infty} \lambda(W_j) = \infty$, since $\lim_{n \rightarrow \infty} \lambda_{n\phi}^0(D) = \infty$. Since the sequence W_j is arbitrary, this concludes the proof of part (1) of Theorem 1.

We now prove the necessity in Part (2) of Theorem 1. As noted before, this also follows from Proposition 2 in [M97]. We provide a proof that does not require any regularity of $b\Omega$. (In this case, $\lambda_{n\phi}(D)$ is defined by the second equality in (2).) We use the fact that compactness of N is equivalent to compactness of Kohn's canonical solution operator $S = \overline{\partial}^* N$, which is in turn equivalent to the following compactness estimates: For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$(5) \quad \|Su\|^2 \leq \epsilon\|u\|^2 + C_\epsilon\|u\|_{-1}^2$$

for all $u \in L^2_{(0,1)}(\Omega)$ (see [FS01], Lemma 1.1).

Let $\beta \in C_0^\infty(D)$ and let $u_n = \beta(z)w^n d\bar{z}$ and $f_n(z, w) = S(u_n)$. Then $f_n(z, w) = g_n(z)w^n$ and $\partial g_n(z)/\partial\bar{z} = \beta(z)$. Plugging this into (5) and using the fact that $\|g_n(z)w^n\|_{-1, \Omega}^2 \leq (1/n^2)\|g_n(z)\|_\Omega^2$, we obtain that there exists $N_\epsilon > 0$ such that $\|g_n(z)w^n\|^2 \leq \epsilon\|\beta(z)w^n d\bar{z}\|^2$ when $n > N_\epsilon$. Therefore,

$$\int_D |g_n(z)|^2 e^{-2(n+1)\phi(z)} dA(z) \leq \epsilon \int_D |\beta(z)|^2 e^{-2(n+1)\phi(z)} dA(z).$$

Duality gives for $u \in C_0^\infty(D)$

$$\begin{aligned}
\int_D |u(z)|^2 e^{2(n+1)\phi(z)} &= \sup\{|\langle u, \beta \rangle|^2; \beta \in C_0^\infty(D), \int_D |\beta|^2 e^{-2(n+1)\phi(z)} \leq 1\} \\
&\leq \sup\{|\langle u_z, g_n \rangle|^2; \int_D |g_n|^2 e^{-2(n+1)\phi} \leq \varepsilon\} \\
&\leq \varepsilon \int_D |u_z|^2 e^{2(n+1)\phi}.
\end{aligned}$$

The middle inequality follows from consideration of the special g_n associated in the previous paragraph to a $\beta \in C_0^\infty(D)$. In view of (2), this concludes the proof of necessity.

We now proof the sufficiency. Since compactness of the $\bar{\partial}$ -Neumann operator is a local property (see [FS01], Lemma 1.2; the direction we need here follows from a simple partition of unity argument) and since by assumption $b\Omega$ is strictly pseudoconvex in a neighborhood of $b\Omega \cap \{w = 0\}$, we need only establish compactness estimates ([FS01], Lemma 1.1) for forms whose support is away from $b\Omega \cap \{w = 0\}$. Moreover, by the interior elliptic regularity of $\bar{\partial} \oplus \bar{\partial}^*$, we need only consider forms whose support is close to $b\Omega$. Choose $\hat{D} \subset\subset D$ such that $b\Omega$ is strictly pseudoconvex on a neighborhood of the part of $b\Omega$ over the outside of \hat{D} .

We work for a moment on $\hat{D} \times S^1$ (S^1 is the unit circle). Denoting the variables on $\hat{D} \times S^1$ by (z, t) , let $L = \partial_z + i\phi_z \partial_t$. We use $||| \cdot |||$ to denote norms on $\hat{D} \times S^1$. We will prove that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(6) \quad |||u|||^2 \leq \varepsilon(|||Lu|||^2 + |||\bar{L}u|||^2) + C_\varepsilon |||u|||_{-1}^2$$

for $u \in C_0^\infty(\hat{D} \times S^1)$. By the assumption on the eigenvalues $\lambda_{n\phi}(D)$, there exists $N_\varepsilon > 0$ such that when $n > N_\varepsilon$,

$$\|v\|^2 \leq \varepsilon \|L_{n\phi}v\|^2, \quad \text{for all } v \in C_0^\infty(\hat{D}).$$

(Note that $\lambda_{n\phi}(\hat{D}) \geq \lambda_{n\phi}(D)$.) Taking conjugates, we obtain that when $n < -N_\varepsilon$,

$$\|v\|^2 \leq \varepsilon \|\bar{L}_{n\phi}v\|^2, \quad \text{for all } v \in C_0^\infty(\hat{D}).$$

Therefore, when $|n| > N_\varepsilon$

$$\|v\|^2 \leq \varepsilon (\|L_{n\phi}v\|^2 + \|\bar{L}_{n\phi}v\|^2), \quad \text{for all } v \in C_0^\infty(\hat{D}).$$

For $u \in C_0^\infty(\hat{D} \times S^1)$, write

$$u = \sum_{n=-\infty}^{\infty} u_n(z) e^{int}$$

where $u_n(z) = (1/2\pi) \int_0^{2\pi} u(z, e^{it}) e^{-int} dt \in C_0^\infty(\hat{D})$. Then

$$Lu = \sum_{n=-\infty}^{\infty} (-L_{n\phi}u_n) e^{int},$$

and

$$\begin{aligned}
\frac{1}{2\pi} \|||u\|||^2 &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \|||u_n\|||^2 = \sum_{n=-\infty}^{\infty} \|u_n\|^2 \\
&\leq \sum_{|n| \leq N_\epsilon} \|u_n\|^2 + \epsilon \sum_{|n| > N_\epsilon} (\|L_{n\phi} u_n\|^2 + \|\bar{L}_{n\phi} u_n\|^2) \\
&= \frac{\epsilon}{2\pi} (\|||Lu\|||^2 + \|||\bar{L}u\|||^2) + \sum_{|n| \leq N_\epsilon} (\|u_n\|^2 - \epsilon(\|L_{n\phi} u_n\|^2 + \|\bar{L}_{n\phi} u_n\|^2)).
\end{aligned}$$

The last sum in the above inequalities is less than or equal to

$$C_\epsilon \sum_{|n| \leq N_\epsilon} \|u_n\|_{-1}^2$$

for some sufficiently large C_ϵ , depending only on ϵ . This is because $\forall n, L_{n\phi}$ and $\bar{L}_{n\phi}$ have a compact inverse (see section 2), which implies $\|u_n\|^2 \leq \epsilon(\|L_{n\phi} u_n\|^2 + \|\bar{L}_{n\phi} u_n\|^2) + C_\epsilon \|u_n\|_{-1}^2$ for a constant C_ϵ . (This is analogous to Lemma 1.1 in [KN65], see also Lemma 1.1 in [FS01].) C_ϵ depends on n , but because we are now only concerned with n 's satisfying $|n| \leq N_\epsilon$, C_ϵ may be chosen depending only on ϵ . The desired inequality (6) now follows from the fact that the last sum above is controlled by $\|||u\|||_{-1}^2$.

We now return to the setting of the Hartogs domain in Theorem 1. For the part of the boundary over \hat{D} , we may use as defining function the function $\rho(z, w) = \frac{1}{2} \log(w\bar{w}e^{2\phi})$. For, say, $0 < r < 1$, the level sets $M_r = \{\rho = -r\}$ are the surfaces $\{|w|^2 = e^{2\phi-2r}\}$. For r fixed, we use coordinates (z, t) on M_r via $(z, t) \leftrightarrow (z, e^{-\phi(z)-r+it})$. Denote by L_1 the usual complex tangential field of type (1,0) given by $\rho_z \partial_w - \rho_w \partial_z$. A computation shows that when restricted to M_r , $2wL_1$ becomes $\partial_z + i\phi_z \partial_t$, which is the operator L considered in the previous paragraph. Let now u be a smooth function supported above \hat{D} and sufficiently close to $b\Omega$. Denote by $d\sigma_r$ the surface measure on M_r . Using that dV in \mathbb{C}^2 is comparable to $d\sigma_r dr$ (on $\text{supp}(u)$), and $d\sigma_r$ is comparable to $dV(z) dt$, uniformly in r , we obtain from (6)

$$\begin{aligned}
\|u\|^2 &= \int_{\Omega} |u|^2 \simeq \int_0^1 \left(\int_{M_r} |u|^2 d\sigma_r \right) dr \\
&\lesssim \epsilon \int_0^1 \left(\int_{M_r} (|Lu|^2 + |\bar{L}u|^2) d\sigma_r \right) dr + C_\epsilon \int_0^1 \|u\|_{-1, M_r}^2 dr \\
&\lesssim \epsilon \int_0^1 \left(\int_{M_r} (|L_1 u|^2 + |\bar{L}_1 u|^2) d\sigma_r \right) dr + C_\epsilon \|u\|_{-1}^2 \\
&\lesssim \epsilon (\|L_1 u\|^2 + \|\bar{L}_1 u\|^2) + C_\epsilon \|u\|_{-1}^2.
\end{aligned}$$

Here, as usual, \lesssim indicates ‘‘less than or equal to, up to a constant factor that is independent of ϵ ’’. Let now $\alpha = a_1 d\bar{z} + a_2 d\bar{w} \in C_{(0,1)}^\infty(\bar{\Omega}) \cap \text{Dom } \bar{\partial}^*$, with support above \hat{D} and close to $b\Omega$. Changing C_ϵ if necessary, we get from the estimate above

$$\|\alpha\|^2 \leq \epsilon(\|L_1\alpha\|^2 + \|\overline{L}_1\alpha\|^2) + C_\epsilon\|\alpha\|_{-1}^2,$$

where L_1 and \overline{L}_1 act componentwise on forms, as usual.

We next invoke maximal estimates ([D78], Théorème 3.1): in \mathbb{C}^2 , $\|L_1\alpha\|^2 + \|\overline{L}_1\alpha\|^2$ is controlled by $\|\overline{\partial}\alpha\|^2 + \|\overline{\partial}^*\alpha\|^2$. (Actually, the statement in [D78] includes the term $\|\alpha\|^2$, but this term is now well known to be bounded by $\|\overline{\partial}\alpha\|^2 + \|\overline{\partial}^*\alpha\|^2$; alternatively, we may absorb it into the left hand side.) The result of applying the maximal estimates is (again, C_ϵ may have to be increased):

$$\|\alpha\|^2 \leq \epsilon(\|\overline{\partial}\alpha\|^2 + \|\overline{\partial}^*\alpha\|^2) + C_\epsilon\|\alpha\|_{-1}^2.$$

This is the required compactness estimate. The proof of Theorem 1 is complete.

Remark 4. The assumption in Theorem 1 that Ω is strictly pseudoconvex near the boundary of the base is not essential. It suffices for example that the boundary is of finite type ([DA93]) near points of $b\Omega \cap \{w = 0\}$. One can then replace W by the (Euclidean) closure of $\text{int}_f(W)$ in the above proofs (compare [F99]). This set will be relatively compact in D because $\Delta\phi$ vanishes to infinite order at fine interior points of W (see [He69], Corollary 10.5 or Theorem 10.14). We leave the details to the reader.

5. APPENDIX

In this section, we show that on the domains considered in Theorem 1, property (P) and property (\tilde{P}) are actually equivalent.

We first recall the definition of property (\tilde{P}) by McNeal in [McN01]. A compact set K in \mathbb{C}^n is said to satisfy property (\tilde{P}) if for any $M > 0$, there exists a neighborhood U of K and $g \in C^2(U)$ such that

- (1) $|\langle \partial g, X \rangle|^2 \leq L_g(X)$;
- (2) $L_g(X) \geq M|X|^2$.

Here $\langle \cdot, \cdot \rangle$ is the pairing between a form and a vector and $L_g(X) = \overline{\partial}\partial g(X, \overline{X})$. (1) is equivalent to $-e^{-g}$ being plurisubharmonic in U (see the discussion in [McN01]).

Lemma 6. *Let Ω be a smooth bounded complete pseudoconvex Hartogs domain in \mathbb{C}^2 . Assume that $b\Omega$ is strictly pseudoconvex at the base. Then $b\Omega$ satisfies property (P) if and only if it satisfies property (\tilde{P}) .*

Proof. It is easy to see that property (P) always implies property (\tilde{P}) ([McN01]): if λ is the function in the definition of property (P) , it suffices (modulo a normalization) to consider the function $g = e^\lambda$. The other direction follows by combining Lemma 7 and Lemma 8 below. \square

Let $\Omega = \{(z, w); z \in D, |w| < e^{-\phi}\}$. Then $\Delta\phi \geq 0$ and the weakly pseudoconvex points correspond to the set of base points $W = \{z \in D | \Delta\phi = 0\}$ for. Note that $W \subset\subset D$.

Lemma 7. *Let K be a compact subset of \mathbb{C} . Then K satisfies property (P) if and only if it satisfies property (\tilde{P}).*

Proof. We only have to show that property (\tilde{P}) implies property (P). In light of Proposition 5, it suffices to show that for any open sets U_j such that $K \subset\subset U_{j+1} \subset\subset U_j$ and $\bigcap_{j=1}^{\infty} \overline{U_j} = K$, $\lim_{j \rightarrow \infty} \lambda(U_j) = \infty$.

For any $M > 0$, there exists a neighborhood U of K and $g \in C^2(U)$ such that $|g_z|^2 \leq g_{z\bar{z}}$ and $g_{z\bar{z}} \geq M$ on U . Assume that j_0 is sufficiently large so that $U_{j_0} \subset\subset U$. It follows from an easy integration by parts that

$$\int_{U_{j_0}} |u_z - \frac{1}{2}g_z u|^2 dA = \frac{1}{2} \int_{U_{j_0}} g_{z\bar{z}} |u|^2 dA + \int_{U_{j_0}} |u_{\bar{z}} + \frac{1}{2}g_{\bar{z}} u|^2 dA$$

for any $u \in C_0^\infty(U_{j_0})$. The left hand side of the above equation is bounded from above by $3\|u_z\|^2 + \frac{3}{8}\|g_z u\|^2$ while the right hand side is bounded from below by $\frac{1}{2} \int g_{z\bar{z}} |u|^2 dA$. Therefore,

$$\int_{U_{j_0}} |u_z|^2 dA \geq \frac{1}{24} \int_{U_{j_0}} g_{z\bar{z}} |u|^2 dA \geq \frac{M}{24} \int_{U_{j_0}} |u|^2 dA.$$

Hence $\lambda(U_j) \geq \lambda(U_{j_0}) \geq M/6$ when $j \geq j_0$. This concludes the proof of Lemma 7. \square

Lemma 8. *Assumptions as in Lemma 6. Then*

1. $b\Omega$ satisfies property (P) if and only if W satisfies property (P).
2. $b\Omega$ satisfies property (\tilde{P}) if and only if W satisfies property (\tilde{P}).

Proof. Part (1) may be found in [Si87], page 310.

To prove (2), first note that if W satisfies property(\tilde{P}), it satisfies property(P) (Lemma 7), hence so does $b\Omega$, by part(1). But then $b\Omega$ also satisfies property(\tilde{P}), by the discussion above.

The proof of the other direction is completely analogous to the proof of the corresponding direction in (1). We are indebted to Nessim Sibony for a private communication ([Si95]) on the details of the argument in [Si87]. Fix $M > 0$. Let g be the corresponding plurisubharmonic function from the definition of property(\tilde{P}). Replacing g by $(1/2\pi) \int_0^{2\pi} g(z, we^{i\theta}) d\theta$, we may assume that g is invariant under rotations in the w variable. Consider $h(z) := g(z, e^{-\phi(z)})$, defined in a neighborhood of \overline{D} . Then, for a sufficiently small neighborhood U of W

$$(7) \quad h_{z\bar{z}} \geq M ; |h_z|^2 \leq h_{z\bar{z}}.$$

This is a matter of computation. This computation can be somewhat simplified by first observing that the function $g_1(z, w) := g(z, e^w)$, defined in a neighborhood of the set $\{(z, w) \in \mathbb{C}^2 | z \in D, w + \bar{w} = -2\phi(z)\}$, also satisfies (1) and (2) in the definition of property (\tilde{P}), with M replaced by, say, $\tilde{M} = (\min\{e^{-2|\phi(z)|-1} | z \in U\})M$, where U is a suitable neighborhood of W (after shrinking the neighborhood where g_1 is defined, independently of M). Now $h(z) = g_1(z, -\phi(z))$; also note that since g is invariant

under rotations in the w variable, g_1 is independent of the imaginary part of w , that is, $(g_1)_w \equiv (g_1)_{\bar{w}}$. It follows that

$$h_z = (g_1)_z - 2(g_1)_w \phi_z = \langle \partial g_1, X \rangle$$

and

$$h_{z\bar{z}} = L_{g_1}(X) - 2(g_1)_w \phi_{z\bar{z}},$$

where $X = (1, -2\phi_z)$. Consequently, (7) is satisfied at points of W (where $\phi_{z\bar{z}} = 0$), up to replacing M by \tilde{M} . Rescaling h (for example, replacing h by $h/2$) allows one to conclude (7) for z in a small enough neighborhood of W (by continuity). \square

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