Improved approximation algorithms for minimum power covering problems

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Abstract

Given an undirected graph with edge costs, the **power of a node** is the maximum cost of an edge incident to it, and the **power of a graph** is the sum of the powers of its nodes. Motivated by applications in wireless networks, we consider two network design problems under the power minimization criteria. In both problems we are given a graph $G = (V, E)$ with edge costs and a set $T \subseteq V$ of terminals. The goal is to find a minimum power edge subset $F \subseteq E$ such that the graph $H = (V, F)$ satisfies some prescribed requirements. In the Min-Power Edge-Cover problem, H should contain an edge incident to every terminal. Using the Iterative Randomized Rounding (IRR) method, we give an algorithm with expected approximation ratio 1.41; the ratio is reduced to $73/60 < 1.217$ when T is an independent set in G. In the case of unit costs we also achieve ratio $73/60$, and in addition give a simple efficient combinatorial algorithm with ratio $5/4$. For all these NP-hard problems the previous best known ratio was $3/2$. In the related Min-Power Terminal Backup problem, H should contain a path from every $t \in T$ to some node in $T \setminus \{t\}$. We obtain ratio $3/2$ for this NP-hard problem, improving the trivial ratio of 2.

Keywords: Approximation algorithms; Iterative randomized rounding; Minimum power; Edge-cover; Terminal backup

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[1] Partially supported by NSF grant number 1540547.

1. Introduction

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. Conversely, given a communication network, the power required at v only depends on the farthest node that is reached directly by v. This is in contrast with wired networks, in which every pair of stations that need to communicate directly incurs a cost. Thus the minimal power $p(v)$ of a node v equals the largest cost of an edge incident to v in the communication network. The first work under the minimum power model is from 1989 [8]. For a sample of other works under this model see for example [1, 5, 18, 24, 28, 9, 6, 7, 9, 10, 17, 20, 21, 23, 26, 27, 30, 31, 32, 25, 19].

Definition 1.1. *Let $H = (V, F)$ be a graph with edge-costs $\{c(e) : e \in F\}$. For $v \in V$, the* **power** $p(v) = p_H(v) = p_F(v)$ **of** *v in H (w.r.t. c) is the maximum cost of an edge in F incident to v (or zero, if no such edge exists), i.e., $p(v) = p_F(v) = \max\limits_{vu \in F} c(vu)$. The power of H is the sum of the powers of its nodes, namely, $p(H) = p(F) = \sum_{v \in V} p_F(v)$.*

All the graphs are assumed to be undirected, unless stated otherwise. In our problems, the input is a graph $G = (V, E)$ with edge costs $\{c(e) : e \in E\}$ and a subset $T \subseteq V$ of terminals; the goal is to find a minimum power subgraph $H = (V, F)$ of G that satisfies some prescribed properties. We refer the reader to a recent survey [29] on such problems. We consider the min-power variant of two classic problems, Edge-Cover and Terminal-Backup, defined below.

Definition 1.2. *For a graph $H = (V, F)$ and a set $T \subseteq V$ of terminals, we say that F (or H) is:*

- *a T-**cover** if every $t \in T$ has some edge in F incident to it (equivalently, no connected component of H is a single terminal);*
- *a T-**backup** if every $t \in T$ has a path to some other node in T (equivalently, no connected component of H contains a single terminal).*

MIN-POWER EDGE-COVER Here F should be a T-cover, namely, every $t \in T$ has some edge in F incident to it.

MIN-POWER TERMINAL BACKUP Here F should be a T-backup, namely, every $t \in T$ has a path to some other node in T.

For illustration of an application of the MIN-POWER EDGE-COVER problem, suppose we have two sets A, B of stations. The stations in A can communicate via an existing wired infrastructure, while each station in B should have a wireless communication with some station in A. We want to assign energy levels to the stations while minimizing the total energy. This is modeled as a MIN-POWER EDGE-COVER problem in a bipartite graph with terminal set $T = B$.

The min-cost versions (where one seeks to minimize $c(F) = \sum_{e \in F} c(e)$) of these problems can be solved in polynomial time see [11] and [2], respectively. MIN-COST EDGE COVER is among the most basic problems in combinatorial optimization and theoretical computer science, see for example the book [33]. The MIN-COST TERMINAL BACKUP problem was also widely studied, c.f. [2, 13, 3]. In the case $T = V$ the problems coincide; the resulting min-power problem is still NP-hard by a standard reduction from SET COVER; note that this implies that the min-power versions of both problems are NP-hard. Moreover, the MIN-POWER EDGE-COVER problem is APX-hard even if T is an independent set in the input graph G and all costs are equal to 1 [18].

For each of these problems, any inclusion-minimal solution is a forest, since removing any edge from any cycle keeps the solution feasible and does not increase the objective function. It is known that if F is a forest then $c(F) \leq p(F) \leq 2c(F)$. This implies that both problems admit ratio 2, by simply computing an optimal min-cost solution.

For MIN-POWER EDGE-COVER the trivial ratio 2 was improved to 1.5 in [22]. No better ratio was known even for the case when T is an independent set in G and all costs are equal to 1. We improve this as follows.

Theorem 1.3. MIN-POWER EDGE-COVER *admits a polynomial time algorithm with expected approximation ratio* 1.41. *If T is an independent set in G then the ratio can be reduced to* $73/60 < 1.217$.

The algorithm in Theorem 1.3 uses the Iterative Randomized Rounding (IRR) method. We also use a method of analyzing the best of two algorithm using a convex combination of their results; we have seen this technique in [15].

In the case of unit costs we show a simple approximation ratio preserving reduction to the case when T is an independent set, thus obtaining for this case ratio $73/60 < 1.217$. In addition, we use a different method to obtain an efficient combinatorial approximation algorithm with good ratio.

Theorem 1.4. Min-Power Edge-Cover *with unit costs admits a polynomial time algorithm with expected approximation ratio* $73/60 < 1.217$*. The problem also admits a* $5/4$*-approximation algorithm with running time* $O(n^3)$*.*

We also improve the trivial ratio 2 for Min-Power Terminal Backup.

Theorem 1.5. Min-Power Terminal Backup *admits a polynomial time algorithm with approximation ratio* 1.5*.*

The proof of the latter theorem uses the idea of the 1.5-approximation algorithm in [22] for Min-Power Edge-Cover, but the details are more involved.

We now briefly survey some work where the IRR method is used. This method is due to Byrka, Grandoni, Rothvoß and Sanita [4], that gave a $\ln 4 + \epsilon < 1.39$ approximation for the Min-Cost Steiner Tree problem. This is currently the best ratio known for the problem. Goemans, Olver, Rothvoß and Zenklusen [14] gave faster and simpler $\ln 4 + \epsilon$ approximation for the same problem, and also obtained a better ratio $73/60$ for quasi-bipartite graphs. Grandoni [16] used the IRR method to give the currently best known ratio 1.91 for the Min-Power Steiner Tree problem. Our paper has similarities with [16] including a Harmonic potential function, and two main differences: (i) it is technically easier (for us) to cover terminals than to cover all cuts separating terminals as in [16]; (ii) we combine iterative randomized rounding with another algorithm, since by itself, iterative randomized rounding fails to improve the ratio $3/2$ in some cases.

This paper is organized as follows. Theorem 1.3 is proved in Sections 2 and 3; in Section 2 we formulate the hypergraphic LP-relaxation for the problem, describe the algorithm, and prove the approximation ratio assuming that a specific lemma (Lemma 2.6) holds; this lemma is proved in Section 3. Theorems 1.4 and 1.5 are proved in Sections 4, and 5, respectively. Finally, in Section 6 we describe a more efficient version of our algorithm from Section 2.

2. Algorithm for Min-Power Edge-Cover (Theorem 1.3)

A **star** is a rooted tree R such that only its root r, called the **center**, may have degree ≥ 2. Note that any inclusion-minimal T-cover F is a collection of disjoint stars, as if F has a path of length three, then the middle edge e of this path can be removed and $F \setminus \{e\}$ remains an T-cover.

For $S \subseteq T$ let π_S be the minimum power of a star R_S that contains S ($\pi_S = \infty$ if no such star exists). Note that given S, both R_S and π_S can be computed in polynomial time by "guessing" the center of R_S. For an integer $k \geq 1$ let $\mathcal{T}_k = \{S \subseteq T : |S| \leq k\}$. We say that a subfamily $\mathcal{T} \subseteq \mathcal{T}_k$ is a **k-restricted T-cover** if the union of the sets in $\mathcal{T}$ is T; the power of $\mathcal{T}$ is defined to be $p(\mathcal{T}) = \sum_{S \in \mathcal{T}} \pi_S$. In what follows we denote by $t = |T|$ the number of terminals and by $n = |V(G)|$ the number of nodes in G.

The "hypergraphic" linear program $LP_k(T)$ below has a variable x_S for every $S \in \mathcal{T}_k$, and it is a relaxation for the problem of finding a k-restricted T-cover of minimum power.

$$\begin{array}{lll} \min & \displaystyle\sum_{S \in \mathcal{T}_k} \pi_S x_S & \\ \text{s.t.} & \displaystyle\sum_{S \in \mathcal{T}_k, S \ni v} x_S \geq 1 & \forall v \in T \\ & x_S \geq 0 & \forall S \in \mathcal{T}_k \end{array}$$

It is easy to see that $LP_k(T)$ can be solved in polynomial time for any constant k. Let us call a feasible solution x to $LP_k(T)$ **irreducible** if no coordinate of x can be lowered while keeping feasibility.

Lemma 2.1. *Let x be an irreducible feasible solution to $LP_k(T)$. Then $\sum_{S \in \mathcal{T}_k} x_S \leq n$, and $\mathcal{T}_k$ with probabilities $\Pr[S] = x_S/n$ for $S \neq \emptyset$ and $\Pr[\emptyset] = 1 - \sum_{S \in \mathcal{T}_k} x_S/n$ is a sample space, in which $\Pr[\{S \in \mathcal{T}_k : S \ni v\}] \geq 1/n$ holds for any $v \in T$.*

Proof. Since x is irreducible, for any $S \in \mathcal{T}_k$ with $x_S > 0$ there exists $v \in S$ such that the inequality of v in $LP_k(T)$ is tight. For every $S \in \mathcal{T}_k$ with $x_S > 0$ choose one such node v_S. Let $W = \{v_S : x_S > 0, S \in \mathcal{T}_k\}$ be the set of chosen nodes, and note that $W \subseteq T$. Then

$$\sum_{S \in \mathcal{T}_k} x_S \leq \sum_{v \in W} \sum_{S \in \mathcal{T}_k, S \ni v} x_S \leq \sum_{v \in W} 1 \leq |W| \leq n \ .$$

This implies that $\Pr[\emptyset] = 1 - \sum_{S \in \mathcal{T}_k} x_S/n \geq 0$ and thus we have a sample space. Furthermore, $\Pr[\{S \in \mathcal{T}_k : S \ni v\}] = \sum_{S \in \mathcal{T}_k, S \ni v} x_S/n \geq 1/n$, by the constraint of v in $LP_k(T)$. $\square$

The following lemma provides a (tight) bound on the ratio between the power of an optimal T-cover and a k-restricted T-cover.

Lemma 2.2 ([22])**.** *For any T-cover F there exists a k-restricted T-cover $\mathcal{T}$ of power $p(\mathcal{T}) \leq (1 + 1/k)p(F)$.*

We run two algorithms and take the best of the two. The first algorithm is the 3/2-approximation algorithm of Kortsarz & Nutov [22]; we call it the KN-Algorithm.

Algorithm 1: KN-ALGORITHM$(G = (V, E), c, T)$

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for all $u, v \in T$ (possibly $u = v$) compute a min-power $\{u, v\}$-cover J_{uv}
let (T, E') be a complete graph with all loops and edge costs $c_{uv} = p(J_{uv})$ for all $u, v \in T$
compute a minimum cost T-cover $J' \subseteq E'$
return $J = \bigcup_{uv \in J'} J_{uv}$

The second algorithm is an Iterative Randomized Rounding algorithm, abbreviated by IRR-Algorithm. For previous applications of this type of algorithms see [4, 14] for the MIN-COST STEINER TREE problem, and [16] for the MIN-POWER STEINER TREE problem.

Algorithm 2: IRR-ALGORITHM$(G = (V, E), c, T, k)$

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initialize $J \leftarrow \emptyset$
while $T \neq \emptyset$ **do**
  compute an irreducible optimal solution x for $LP_k(T)$ and sample one set $S \in \mathcal{T}_k$ with probabilities as in Lemma 2.1
  $T \leftarrow T \setminus S$, $J \leftarrow J \cup R_S$
return J

Note that in every iteration, the set of terminals may change. In such a case, the IRR-Algorithm solves a new LP with respect to the new set of terminals. To ensure polynomial time, after $2n \ln n$ iterations the while-loop is terminated, and we add to J a solution for the residual problem computed by the KN-Algorithm. The following lemma shows that the expected loss in the approximation ratio incurred by such modification is negligible.

Lemma 2.3. *In every iteration, every $v \in T$ is hit with probability at least $1/n$. The probability that $T \neq \emptyset$ after $2n \ln n$ iterations is at most $1/n$. The expected loss in the approximation ratio incurred by stopping the IRR algorithm after $2n \ln n$ iterations is at most $\frac{3}{2n}$.*

Proof. The first statement follows from Lemma 2.1. The probability that after $i = 2n \ln n$ iterations a terminal is not hit is at most $(1 - 1/n)^i \leq 1/n^2$. By the union bound the probability that there exists a non hit terminal is at most $1/n$. Finally, in the case that there exists a non hit terminal, the algorithm has an approximation ratio of $3/2$. Thus the loss in the ratio is at most $\frac{3}{2n}$. $\square$

We now give properties of these algorithms that will enable us to prove the approximation ratio. We say that a star R is a **proper star** if R has at least one terminal and, if R has at least two edges, then all the leaves of R are terminals (a star with one edge may have one terminal, that may be the leaf or the center). Fix some proper star R with center r. Note that if R has a single edge then r can be the unique terminal in R. Denote the leaves of R by $v_1, v_2, \ldots, v_q$ arranged by non-increasing edge costs $c_1 \geq c_2 \geq \ldots \geq c_q$ where $c_j = c(rv_j)$ and assume that $c_1 > 0$. Note that $p(R) = c_1 + c(R) = c_1 + \sum_{j=1}^{q} c_j$. Let $\psi(R)$ be defined by:

$$\psi(R) = \begin{cases} c_3 + c_5 + \cdots + c_q & q \geq 3 \text{ odd} \\ c_3 + c_5 + \ldots + c_{q-1} & q \geq 4 \text{ even}, r \notin T \\ c_3 + c_5 + \ldots + c_{q-1} + c_q & q \geq 4 \text{ even}, r \in T \end{cases}$$

Here $\psi(R) = 0$ if $q \in \{1, 2\}$, except that $\psi(R) = c_2$ if $q = 2$ and $r \in T$.

The following lemma is proved in [22], but we provide a proof-sketch for completeness of exposition.

Lemma 2.4 ([22])**.** *Let R be a proper star as above. Then there exists a 2-restricted cover $\mathcal{T}$ of the terminals in R such that $p(\mathcal{T}) \leq p(R) + \psi(R) \leq \frac{3}{2} p(R)$.*

Proof. It is not hard to verify that the following $\mathcal{T}$ is as required:

$$\begin{array}{ll}
\mathcal{T} = \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{q-2}, v_{q-1}\}, \{v_q\}\} & q \text{ odd}, \quad r \notin T \\
\mathcal{T} = \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{q-2}, v_{q-1}\}, \{v_q, r\}\} & q \text{ odd}, \quad r \in T \\
\mathcal{T} = \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{q-3}, v_{q-2}\}, \{v_{q-1}, v_q\}\} & q \text{ even}, \quad r \notin T \\
\mathcal{T} = \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{q-3}, v_{q-2}\}, \{v_{q-1}\}, \{v_q, r\}\} & q \text{ even}, \quad r \in T
\end{array}$$

It is also not hard to see that $\psi(R) \leq \frac{1}{2}p(R)$. □

Assume for a moment that proper star R as above contains all terminals and is an optimal solution to our problem. Then $p(R) + \psi(R)$ bounds the solution value produced by the KN-Algorithm. We will show later that the expected solution value produced by the IRR-Algorithm is bounded by $p(R) + \phi(R)$ where

$$\phi(R) = \begin{cases} \displaystyle\sum_{j=1}^{q} c_j/j & q \geq 1, V(R) \subseteq T \\ \displaystyle\sum_{j=2}^{q} c_j/j & q \geq 1, r \notin T \\ 0 & \text{otherwise } (q = 1,\ V(R) \cap T = \{r\}) \end{cases}$$

The function $\phi(R)$ is built so that the proof of Lemma 2.6 to follow holds; we note that Harmonic functions are also used in [16] and [14].

If we know that our optimal solution is just one star R, then by taking the best outcome of the two algorithms, the (expected) value of the produced solution will be $p(R) + \min\{\psi(R), \phi(R)\}$. In the case of many stars, we take a convex combination of the two algorithms: KN-Algorithm with probability $\theta = 2/3$ and IRR-Algorithm with probability $1-\theta = 1/3$. Since any inclusion-minimal solution is a collection of node-disjoint proper stars, we conclude that the (expected) approximation ratio of the convex combination algorithm is bounded by the maximum possible value of

$$\frac{\theta(p(R) + \psi(R)) + (1-\theta)(p(R) + \phi(R))}{p(R)} = 1 + \frac{1}{3} \cdot \frac{2\psi(R) + \phi(R)}{p(R)}$$

over all the stars R (this assumes that, as shown in Lemma 2.7 below, the expected power of the output of the IRR-Algorithm is $p(R) + \phi(R)$).

For a proper star R as above let us denote (with some abuse of notation) $p(q) = p(R)$, $\psi(q) = \psi(R)$, and $\phi(q) = \phi(R)$. Then the expected approximation ratio of the convex combination algorithm is bounded by $\max_{q\geq 1} \rho(q)$, where

$$\rho(q) = 1 + \frac{1}{3} \max_{c_1 \geq \cdots \geq c_q \geq 0, c_1 > 0} \frac{2\psi(q) + \phi(q)}{p(q)} .$$

We will show later that:

Lemma 2.5. $\rho(q) \leq 1\frac{73}{180} < 1.4056$.

Let $\Phi(R) = p(R) + \phi(R)$ and $\Psi(R) = p(R) + \psi(R)$. It is convenient to also have $\Phi(R) = 0$ if the star R has only a center and no leaves (this is not a proper star, and has $p(R) = 0$). The next lemma, to be proved in Section 3, is the heart of the proof of Theorem 1.3.

Lemma 2.6. *Consider an iteration of the IRR-Algorithm. Let R be a proper star at the beginning of the iteration and let R' be a star obtained from R by removing the leaves of R that are terminals covered at the iteration, with one exception: if only the center of R is an uncovered terminal among $V(R)$ after the iteration, we keep in R' the leaf closest to the center (this means that, unless all the terminals of R are covered, R' remains a proper star). Then $\Phi(R) - E[\Phi(R')] \geq p(R)/n$.*

For a collection $\mathcal{R}$ of stars let $\Phi(\mathcal{R}) := \sum_{R\in\mathcal{R}} \Phi(R)$ and $p(\mathcal{R}) := \sum_{R\in\mathcal{R}} p(R)$.

Lemma 2.7. *Let $\mathcal{R} = \{R_S : S \in \mathcal{T}\}$ be a set of stars of a k-restricted (optimal) T-cover $\mathcal{T}$ and J a solution produced by the IRR-Algorithm. Then $E[p(J)] \leq \Phi(\mathcal{R})$.*

Proof. Let T_{i-1} be the set of terminals uncovered at the beginning of iteration i and τ_i^* the expected optimal value of $LP_k(T_{i-1})$. Let $\mathcal{R}_0 = \mathcal{R}$ and for $i \geq 1$ obtain $\mathcal{R}_i$ from $\mathcal{R}_{i-1}$ by taking, for each proper star in $R \in \mathcal{R}_{i-1}$, the star R' as in Lemma 2.6. Now, note the following:

- $E[p(J)] \leq \sum_{i\geq 1} \tau_i^*/n$, since after solving $LP_k(T_{i-1})$ at iteration $i \geq 1$, each star R_S is selected with probability x_S/n.
- $\tau_i^* \leq E[p(\mathcal{R}_{i-1})]$ at iteration $i \geq 1$, since the stars in $\mathcal{R}_{i-1}$ cover T_{i-1} while τ_i^* is the expected optimal value of $LP_k(T_{i-1})$.
- $E[p(\mathcal{R}_{i-1})]/n \leq E[\Phi(\mathcal{R}_{i-1}) - \Phi(\mathcal{R}_i)]$ at iteration $i \geq 1$, by Lemma 2.6.

Combining we get that the expected power of J is bounded by:

$$E[p(J)] \leq \sum_{i\geq 1} \tau_i^*/n \leq \sum_{i\geq 1} E[p(\mathcal{R}_{i-1})]/n \leq \sum_{i\geq 1} E[\Phi(\mathcal{R}_{i-1}) - \Phi(\mathcal{R}_i)] = \Phi(\mathcal{R})$$

The last equality holds since the sum is telescopic and since $\Phi(\mathcal{R}_0) = \Phi(\mathcal{R})$ is not a random variable. $\square$

Let J_{KN} and J_{IRR} be the outputs of the KN-Algorithm and the IRR-Algorithm, respectively. Let $\mathcal{R}$ and $\mathcal{R}_k$ be optimal and k-restricted optimal set of stars that cover T, respectively. Then $p(\mathcal{R}_k) \leq (1 + 1/k)p(\mathcal{R})$, by lemma 2.2. As was mentioned, in [22] it is proved that $p(J_{KN}) \leq \Psi(\mathcal{R})$. By Lemma 2.7, $p(J_{IRR}) \leq \Phi(\mathcal{R})$. Combining we get that the power of the solution produced by the convex combination of the two algorithms is bounded by

$$\theta p(J_{KN})+(1-\theta)p(J_{IRR}) \leq \theta\Psi(\mathcal{R})+(1-\theta)\Phi(\mathcal{R}_k) \leq \left(1+\frac{1}{k}\right)(\theta\Psi(\mathcal{R})+(1-\theta)\Phi(\mathcal{R}))$$

From Lemma 2.5 we conclude that $\theta\Psi(\mathcal{R}) + (1-\theta)\Phi(\mathcal{R}) \leq 1.4056p(\mathcal{R})$ for $\theta = 2/3$. Consequently, we get that for $\theta = 2/3$ and constant k large enough

$$\theta p(J_{KN}) + (1-\theta)p(J_{IRR}) \leq 1.41p(\mathcal{R}) = 1.41 \cdot \mathsf{opt} \ .$$

To complete the proof of the 1.41 approximation ratio it only remains to prove Lemmas 2.5 and 2.6; Lemma 2.5 is proved below, while Lemma 2.6 is proved in the next section.

For the proof of Lemma 2.5 we bound the function $h(q) = 3(\rho(q) - 1)$, so $\rho(q) = 1 + \frac{1}{3}h(q)$. For simplicity of notation let us write

$$h(q) = \frac{2\psi(q) + \phi(q)}{p(q)} \quad \text{meaning} \quad h(q) = \max_{c_1\geq c_2\geq\cdots\geq c_q, c_1>0} \frac{2\psi(q) + \phi(q)}{p(q)} \ .$$

Note that Lemma 2.5 follows immediately from the following lemma:

Lemma 2.8. $h(q) \leq \frac{73}{60}$.

Proof. Let us consider the cases $q = 1, 2, 3, 4, 5$.

1. $\psi(1) = 0$, $\phi(1) \leq c_1$ and $p(1) = 2c_1 > 0$, hence $h(1) = 1/2$.
2. $\psi(2) = c_2$, $\phi(2) = c_1 + c_2/2$, and $p(2) = 2c_1 + c_2$, hence $h(2) \leq \frac{c_1+\frac{5}{2}c_2}{2c_1+c_2} \leq \frac{7}{6}$.

3. We have $h(3) = \frac{2c_3+c_1+c_2/2+c_3/3}{2c_1+c_2+c_3} \leq \frac{23}{24}$.
4. We have $h(4) = \frac{2c_3+2c_4+c_1+c_2/2+c_3/3+c_4/4}{2c_1+c_2+c_3+c_4}$, and this can be verified to be at most $\frac{73}{60}$ by expanding and using $c_1 \geq c_2 \geq c_3 \geq c_4$ (we get equality when $c_1 = c_2 = c_3 = c_4$).
5. We have $h(5) = \frac{2c_3+2c_5+c_1+c_2/2+c_3/3+c_4/4+c_5/5}{2c_1+c_2+c_3+c_4+c_5}$, and this can be verified to be at most $\frac{73}{60}$ by expanding and using $c_1 \geq c_2 \geq c_3 \geq c_4 \geq c_5$.

For $q > 5$, we use induction on q. For even $q > 4$, we must prove: $60(2(c_3 + c_5 + \cdots c_{q-1} + c_q) + \sum_{j=1}^{q} c_j/j) \leq 73(c_1 + \sum_{j=1}^{q} c_j)$, which follows from summing up the inductive hypothesis: $60(2(c_3 + c_5 + \cdots c_{q-3} + c_{q-2}) + \sum_{j=1}^{q-2} c_j/j) \leq 73(c_1 + \sum_{j=1}^{q-2} c_j)$ and the inequalities $60c_q \leq 60c_{q-2}$, $60c_{q-1} \leq 60c_{q-2}$, $60c_q(1 + 1/q) \leq 73c_q$, and $60c_{q-1}(1 + 1/(q-1)) \leq 73c_{q-1}$.

For odd $q > 5$, we must prove: $60(2(c_3 + c_5 + \cdots + c_q) + \sum_{j=1}^{q} c_j/j) \leq 73(c_1 + \sum_{j=1}^{q} c_j)$, which follows from summing up the inductive hypothesis: $60(2(c_3+c_5+\cdots+c_{q-2})+\sum_{j=1}^{q-2} c_j/j) \leq 73(c_1+\sum_{j=1}^{q-2} c_j)$ and $(120+60/q)c_q + c_{q-1}60/(q-1) \leq 73(c_{q-1} + c_q)$. □

In the case when T is an independent set in G, no star has center in T. In this case, we simply run the IRR-Algorithm. The approximation ratio stated for this case in Theorem 1.3 follows from the following lemma.

Lemma 2.9. *If T is an independent set in G then $\Phi(q)/p(q) \leq \frac{73}{60}$.*

Proof. In this case, we have $\Phi(q) = \sum_{j=1}^{q} c_j(1 + 1/j)$. One obtains that $60\Phi(q) \leq 73p(q)$ for $q = 1, 2, 3, 4$ by inspection, using $c_1 \geq c_2 \geq c_3 \geq c_4$. The bound is tight for $q = 4$ and $c_1 = c_2 = c_3 = c_4$. For $q \geq 5$, the bound follows from the fact that $60(1 + 1/j) \leq 73$ for any $j \geq 5$. □

3. Proof of Lemma 2.6

Let us write explicitly the function Φ:

$$\Phi(R) = p(R)+\phi(R) = \begin{cases} c_1 + \sum_{j=1}^{q} c_j(1 + 1/j) & q \geq 1, V(R) \subseteq T \\ \sum_{j=1}^{q} c_j(1 + 1/j) & q \geq 1, r \notin T \\ 2c_1 & \text{otherwise } (q = 1,\ V(R) \cap T = \{r\}) \end{cases}$$

We split the proof into two cases: $r \notin T$ and $r \in T$.

3.1. The case $r \notin T$

Recall that a set-function f on a groundset U is submodular if for any $A \subseteq U$ and $a_j, a_k \in U \setminus A$ we have:

$$\Delta_f(A, \{a_j, a_k\}) := f(A \cup \{a_j\}) + f(A \cup \{a_k\}) - f(A) - f(A \cup \{a_j, a_k\}) \geq 0 \ .$$

We will need the following lemma. We believe this lemma is known, but we failed to find its proof in the literature.

Lemma 3.1. *Let U be a set of items with non-negative weights $\{w(u) : u \in U\}$ and let $z_1 \geq z_2 \geq \cdots \geq z_{|U|}$ be reals. Let $f(\emptyset) := 0$ and for $\emptyset \neq A \subseteq U$ define $f(A) := \sum_{i=1}^{|A|} z_i w(a_i)$, where $a_1, \ldots, a_{|A|}$ is an ordering of A such that $w(a_1) \geq \cdots \geq w\left(a_{|A|}\right)$. Then f is submodular and non-decreasing.*

Proof. Let $A \subseteq U$ and $a_j, a_k \in U \setminus A$. Order the elements in $A \cup \{a_j, a_k\}$ in non-increasing order $a_1, \ldots, a_{|A|+2}$ by the weights $w_1 \geq \cdots \geq w_{|A|+2}$, and suppose w.l.o.g. that this order is $a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_{|A|+2}$. Note that the terms in the sums defining $f(A \cup \{a_k\})$ and $f(A)$ coincide up to the kth term, and this so also for $f(A \cup \{a_j\})$ and $f(A \cup \{a_j, a_k\})$. Then we have:

$$f(A \cup \{a_k\}) - f(A) = \sum_{i=k}^{|A|+2} w_i z_{i-1} - \sum_{i=k+1}^{|A|+2} w_i z_{i-2} = \sum_{i=k}^{|A|+2} w_i z_{i-1} - \sum_{i=k}^{|A|+1} w_{i+1} z_{i-1}$$

$$f(A \cup \{a_j\}) - f(A \cup \{a_j, a_k\}) = \sum_{i=k+1}^{|A|+2} w_i z_{i-1} - \sum_{i=k}^{|A|+2} w_i z_i = \sum_{i=k}^{|A|+1} w_{i+1} z_i - \sum_{i=k}^{|A|+2} w_i z_i$$

Consequently,

$$\begin{aligned} \Delta_f(A, \{a_j, a_k\}) &= \sum_{i=k}^{|A|+2} w_i z_{i-1} - \sum_{i=k}^{|A|+1} w_{i+1} z_{i-1} + \sum_{i=k}^{|A|+1} w_{i+1} z_i - \sum_{i=k}^{|A|+2} w_i z_i \\ &= \sum_{i=k}^{|A|+2} w_i (z_{i-1} - z_i) - \sum_{i=k}^{|A|+1} w_{i+1} (z_{i-1} - z_i) \\ &\geq \sum_{i=k}^{|A|+1} (w_i - w_{i+1})(z_{i-1} - z_i) \geq 0 \end{aligned}$$

This shows that f is submodular. It is easy to see that f is non-decreasing. $\square$

We want to show that $\Phi(R) - E[\Phi(R')] \geq p(R)/n$. Let $\bar{R}$ be the set of leaves of R. The case $\bar{R} = \emptyset$ is obvious hence we assume that $\bar{R} \neq \emptyset$.

In this case, by definition, $\Phi(R) = \sum_{j=1}^{q}(1 + 1/j)c_j$. Therefore if we set in Lemma 3.1 $w_i = c_i$ for every i and and $z_i = 1 + 1/i$ for $1 \leq i \leq q$, then by definition $f(\bar{R}) = \Phi(R)$.

Definition 3.2. *Let $\tilde{H}$ be the random variable of the set of terminals hit in iteration i. For $H \subseteq T$ we denote the probability that $\tilde{H} \cap \bar{R} = H$ by $\Pr[H]$, namely, that H is exactly the set of hit terminals among the nodes of R.*

Denote $\Delta(H) = \Phi(R) - \Phi(R')$; in this case ($r \notin T$), we have $\Delta(H) = f(\bar{R}) - f(\bar{R} \setminus H)$. Consider some arbitrary set $H \subseteq \bar{R}$ of possible terminals that could be hit. The following lemma is a standard consequence of submodularity:

Lemma 3.3. $\Delta(H) \geq \sum_{v \in H} \Delta(\{v\})$.

Proof. We have

$$
\begin{aligned}
\Delta(H) &= f(\bar{R}) - f(\bar{R} \setminus H) = && \text{(As the sum is telescopic)} \\
&= \sum_{\ell=1}^{p} f\left(\bar{R} \setminus \{v_1, \dots v_{\ell-1}\}\right) - f(\bar{R} \setminus \{v_1, \dots v_\ell\}) \geq && \text{(As } f \text{ is submodular)} \\
&\geq \sum_{\ell=1}^{p} f(\bar{R}) - f(\bar{R} \setminus \{v_\ell\}) = \sum_{\ell=1}^{p} \Delta(\{v_\ell\})
\end{aligned}
$$

$\square$

Therefore

$$
\begin{aligned}
E[\Delta(H)] &= \sum_{H \subseteq \bar{R}} \Pr[H]\Delta(H) \geq && \text{(As } \Delta(H) \geq \sum_{\ell} \Delta(v_\ell)) \\
&\geq \sum_{H \subseteq \bar{R}} \left(\Pr[H] \sum_{v \in H} \Delta(\{v\}) \right) = && \text{(By changing summation order)} \\
&= \sum_{v \in \bar{R}} \left(\Delta(\{v\}) \sum_{H \subseteq \bar{R} \mid v \in H} \Pr[H] \right) \\
&= \sum_{v \in \bar{R}} \Delta(v) \Pr[v \text{ is hit}] \geq \sum_{v \in \bar{R}} \Delta(v) \frac{1}{n}
\end{aligned}
$$

To justify the last equality, note that $\sum_{H \subseteq \bar{R} \mid v \in H} \Pr[H] = \Pr[v \text{ is hit}]$ because we sum the probabilities of all sets H that contain v. The last inequality follows from Lemma 2.1, which states that the probability that v is hit is at least $1/n$.

What remains to be proved is that

$$\sum_{v \in \bar{R}} \Delta(v) \geq p(R) \tag{1}$$

We need to measure the change in the potential $\Delta(v_\ell)$ (recall that v_ℓ is the ℓ^{th} child of the star R). Also recall that in the potential function $\Phi(R)$, c_ℓ is multiplied by $(1+1/\ell)$. The addition of v_1 (and its most expensive edge) shifts all indexes by 1. This means that $v_{\ell-1}$ becomes v_ℓ. In the new star with v_1 the coefficient of the edge number ℓ is $1+1/\ell$ and in the star without this edges it was $1/(\ell-1)$. Thus the difference between the coefficients is $-(1/(\ell-1) - 1/\ell)$.

Suppose that we add an edge rv_p, $p \geq 2$. Then the coefficients are shifted only for edges that are $p+1$ smallest or later. This means that the sum will start with $\ell = p+1$. Indeed adding edge number p does not change the location of the $p-1$ first edges. Thus the changes are as follows:

$$\begin{aligned}
\Delta(\{v_1\}) &\geq 2c_1 - \sum_{\ell=2}^{q} c_\ell \left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) \\
\Delta(\{v_2\}) &= c_2\left(1+\frac{1}{2}\right) - \sum_{\ell=3}^{q} c_\ell \left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) \\
&\cdots \\
\Delta(\{v_k\}) &= c_k\left(1+\frac{1}{k}\right) - \sum_{\ell=k+1}^{q} c_\ell \left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) \\
&\cdots \\
\Delta(\{v_q\}) &= c_q\left(1+\frac{1}{q}\right)
\end{aligned}$$

Note that the coefficient of edge k is counted $k-1$ times and thus we get by summing up these equations that:

$$\sum_{k=1}^{q} \Delta(v_k) \geq 2c_1 + \sum_{k=2}^{q} c_k \left(1 + \frac{1}{k} - (k-1)\left(\frac{1}{k-1} - \frac{1}{k}\right)\right) = 2c_1 + \sum_{k=2}^{q} c_k = p(R),$$

ending the proof for the case $r \notin T$.

3.2. The case $r \in T$

Note that the equality $\Phi(R) - \Phi(R') = f(\bar{R}) - f(\bar{R}')$ no longer holds in all the cases, because $\Phi(R) = f(\bar{R}) + c_1$, but this may not hold for $\Phi(R')$. Precisely, the bound $\Delta(H)$ is by definition:

$$\begin{array}{ll} \Delta(H) = f(\bar{R}) - f(\bar{R}') + c_1 & \text{if } r \text{ is hit } (H \ni r) \\ \Delta(H) = f(\bar{R}) - f(\bar{R}') + c_1 - c_1' & \text{if } r \text{ is not hit } (H \not\ni r) \text{ and } \bar{R} \neq H \\ \Delta(H) = f(\bar{R}) + c_1 - 2c_q & \text{if } \bar{R} = H \end{array}$$

Indeed, if r is hit, then c_1 does not appear anymore in $\phi(R')$, since r is no longer a terminal. If r is not hit, its power goes from c_1 to c_1'. In the case $\bar{R} = H$ we get that $\Phi(R) - \Phi(R') = (c_1 - 2c_q) + \sum_{j \geq 1}(1 + 1/j) \cdot c_j$. This is because R' is defined to keep from R only the leaf closest to the center, and therefore $\Phi(R') = 2c_q$.

Corollary 3.4. *If $\bar{R} \neq H$ then $\Phi(R) - \Phi(R') \geq f(\bar{R}) - f(\bar{R}')$. If $\bar{R} = H$ then $\Delta(H) = f(\bar{R}) + c_1 - 2c_q \geq f(\bar{R}) - c_1$.*

We continue with the proof of Lemma 2.6 for the case $r \in T$. We first assume that $\Pr[\bar{R}] \leq 1/n$. Then we have

$$\begin{aligned} E[\Delta(H)] &= \Pr[\bar{R}] \cdot \Delta(\bar{R}) + \sum_{H \neq \bar{R}} \Pr[H] \cdot \Delta(H) \qquad \text{(Corollary 3.4 and } \Pr[\bar{R}] \leq 1/n) \\ &\geq -\frac{1}{n}c_1 + \Pr[\bar{R}]f(\bar{R}) + \sum_{H \neq \bar{R}} \Pr[H]\Delta(H) \quad \text{(By separating } r \text{ from the sum)} \\ &\geq -\frac{1}{n}c_1 + \Pr[\bar{R}]\Delta(\bar{R}) + \sum_{H \not\ni r, H \neq \bar{R}} \Pr[H]\Delta(H) + \sum_{H \ni r} \Pr[H]\Delta(H) \end{aligned}$$

By the definition of Δ we get that $E[\Delta(H)] + \frac{1}{n}c_1$ is at least

$$\Pr[\bar{R}]f(\bar{R}) + \sum_{H \not\ni r, H \neq \bar{R}} \Pr[H](f(\bar{R}) - f(\bar{R} \setminus H)) + \sum_{H \ni r} \Pr[H](c_1 + f(\bar{R}) - f(\bar{R} \setminus H))$$

$$= \sum_{H \not\ni r} \Pr[H](f(\bar{R}) - f(\bar{R} \setminus H)) + \sum_{H \ni r} \Pr[H](c_1 + f(\bar{R}) - f(\bar{R} \setminus H))$$

Lemma 3.3 submodularity implies that the last expression is at least

$$\sum_{H \not\ni r} \sum_{v \in H} \Pr[H] \sum_{v \in H} (f(\bar{R}) - f(\bar{R} \setminus \{v\})) + \sum_{H \ni r} \Pr[H](c_1 + \sum_{v \in H \setminus \{r\}} (f(\bar{R}) - f(\bar{R} \setminus \{v\}))$$

By rearranging terms and applying Lemma 2.3 we get

$$
\begin{aligned}
E[\Delta(H)] &\geq -\frac{1}{n}c_1 + \left(\sum_{v\in\bar{R}}(f(\bar{R}) - f(\bar{R}\setminus\{v\}))\cdot\sum_{H\ni v}\Pr[H]\right) + c_1\cdot\sum_{H\ni r}\Pr[H] \\
&\geq -\frac{1}{n}c_1 + \left(\sum_{v\in\bar{R}}(f(\bar{R}) - f(\bar{R}\setminus\{v\}))\frac{1}{n}\right) + c_1\cdot\frac{1}{n} \\
&= \frac{1}{n}\sum_{v\in\bar{R}}(f(\bar{R}) - f(\bar{R}\setminus\{v\})) \\
&\geq p(R)/n,
\end{aligned}
$$

where the last inequality is as in the case $r \notin T$.

The second case is if $\Pr[\bar{R}] > 1/n$. In this case only the contribution of disjoint events $H = \bar{R}$ and $r \in H$ is taken into account:

$$
\begin{aligned}
E[\Delta(R)] &\geq \Pr[\bar{R}]\Delta(\bar{R}) + \Pr[r \text{ is hit}]\cdot\Delta(\{r\}) && \text{(Corollary 3.4)} \\
&\geq \Pr[\bar{R}]\left(f(\bar{R}) - c_1\right) + \Pr[r \text{ is hit }]\cdot\Delta(\{r\}) && \left(\text{We assume } \Pr[\bar{R}] \geq 1/n\right) \\
&\geq \frac{1}{n}\cdot\left(f(\bar{R}) - c_1\right) + \Pr[r \text{ is hit }]\cdot\Delta(\{r\}) && \text{(Lemma 2.3)} \\
&\geq \frac{1}{n}\cdot\left(f(\bar{R}) - c_1\right) + \Delta(\{r\})/n && \text{(Definition of } \Delta) \\
&= \frac{1}{n}\cdot f(\bar{R}) && \text{(Definition of } f) \\
&\geq \frac{1}{n}p(R).
\end{aligned}
$$

This finishes the proof of Lemma 2.6 and thus the proof of Theorem 1.3 is complete.

4. Min-Power Edge-Cover with unit costs (Theorem 1.4)

Let $E(T)$ denote the set of edges in E that have both endnodes in T. We say that a star S in G is a **proper star** if all the leaves of S are terminals. Let T_S denote the set of terminals in S. Our algorithm for unit costs is as follows.

Algorithm 3: UNIT-COSTS-ALGORITHM($G = (V, E), T$) (ratio 5/4)

1 $F \leftarrow E(T)$, $E \leftarrow E \setminus E(T)$, exclude from T terminals covered by $E(T)$
2 while there is a proper star S in G with $|T_S| \geq 4$ do
  $F \leftarrow F \cup S$, $T \leftarrow T \setminus T_S$, $G \leftarrow G \setminus T_S$
3 compute a solution with the KN-Algorithm (with input the current G and T) and add this solution to F
4 return F

In the case of unit costs, if F is a feasible solution then $p(F \cup E(T)) = p(F)$, since $p_F(v) = 1$ for all $v \in T$. This implies that there exists an optimal solution F such that $E(T) \subseteq F$, and thus step 1 in the algorithm is optimal. Note that after this step T is an independent set in G, and thus we can get ratio 73/60, by the second part of Theorem 1.3. This concludes the first part of Theorem 1.4.

We now prove the second part of Theorem 1.4. We may assume that $V \setminus T$ is an independent set, as edges in $E(V \setminus T)$ do not cover any terminal. Consider an iteration at step 2 when a proper star S with $k \geq 4$ terminals is chosen. Adding S to F increases $p(F)$ by $k+1$ and removing T_S from G reduces the optimum by at least k. Hence it is a $\frac{k+1}{k} \leq 5/4$ local ratio step. We now show that the KN-Algorithm achieves ratio 5/4 for the residual instance.

Lemma 4.1. *In the case of unit costs, if T is an independent set in G and if G has no star with 4 terminals then the KN-Algorithm has ratio 5/4.*

Proof. In [22] the following is proved.
If for any proper star S in G there exists a 2-restricted T-cover $\mathcal{T}$ of T_S such that $p(\mathcal{T}) \leq \alpha p(S)$, then the KN-Algorithm achieves ratio α.

In the case considered in the lemma we have unit costs, the center of S is not a terminal, and S has at most 3 leaves. Let u be the center of S and $\{v_1, \ldots, v_q\}$ the set of leaves of S, where $u \notin T$ and $q \in \{1, 2, 3\}$. Note that $p(S) = q + 1$. If $q = 3$ then we take $\mathcal{T} = \{\{v_1\}, \{v_2, v_3\}\}$; then $p(S) = 4$ and $p(\mathcal{T}) = 2 + 3 = 5$. If $q \in \{1, 2\}$ then we take $\mathcal{T} = \{T(S)\}$ and get $p(S) = q + 1 = p(\mathcal{T})$. In both cases $p(\mathcal{T}) \leq \frac{5}{4} p(S)$, and the lemma follows. □

In the worse time complexity case $|T| = \Theta(n)$, the running time of the algorithm is dominated by the running time of step 3 of the KN-Algorithm, that requires computing a minimum cost T-cover in a complete graph on T. This can be done in time $O(n^3)$ by the algorithm of Edmonds and Johnson [12]. This concludes the proof of Theorem 1.4.

5. Min-Power Terminal Backup (Theorem 1.5)

We reduce Min-Power Terminal Backup to the Min-Cost Edge-Cover problem, that is solvable in polynomial time, c.f., [33]. However, the reduction is not approximation ratio preserving, but incurs a loss of 3/2 in the approximation ratio. That is, given an instance (G, c, T) of Min-Power Terminal Backup, we construct in polynomial time an instance (G', c', T) of Min-Cost Edge-Cover such that:

(i) For any T-cover F' in G' corresponds a T-backup F in G with $p(F) \leq c'(F')$.

(ii) $\mathsf{opt}' \leq 3\mathsf{opt}/2$, where opt' is the minimum cost of a T-cover in (G', c', T).

Hence if F' is an optimal (min-cost) solution to (G', c', T), then

$$p(F) \leq c'(F') = \mathsf{opt}' \leq 3\mathsf{opt}/2 \ .$$

Definition 5.1. *A* **spider** *is a rooted tree such that only its root, called the* **center***, may have degree 3 or more (equivalently, a spider is a subdivision of a star). Given a set T of terminals, we say that a spider S is* T**-proper** *if the set $S \cap T$ of its terminals is the set of leaves of S.*

Given an instance of Min-Power Terminal Backup or Min-Cost Terminal Backup, we may assume that all the terminals have degree 1, namely, each $t \in T$ has a unique edge in G incident to it; this is achieved by a standard reduction of adding for every $t \in T$ a new node t' and an edge tt' of cost 0, and making t' a terminal instead of t (note that this reduction does not work for Min-Power Edge-Cover). Under this assumption, we have the following.

Proposition 5.2. *Let F be an inclusion minimal T-backup. Then any connected component C of the graph $H = (V, F)$ is a T-proper spider.*

Proof. Clearly, F is a tree and every leaf of C is a terminal. If this tree has two nodes of degree at least three, than removing an edge on the path between these two nodes results in a valid T-backup. □

We now define a certain decomposition of spiders, similar to the decompositions of stars in [22].

Definition 5.3. *Let S be a spider. A collection $\mathcal{D}$ of paths between the leaves of S such that every leaf belongs to some path is called a* **2-decomposition** *of S. The power $p(\mathcal{D}) = \sum_{S_j \in \mathcal{D}} p(S_j)$ of $\mathcal{D}$ is the sum of the powers of its paths.*

Lemma 5.4. *Any spider S admits a 2-decomposition $\mathcal{D}$ with $p(\mathcal{D}) \le \frac{3}{2}p(S)$.*

Proof. If S is a path then the statement is obvious, so assume that S has at least 3 leaves. Let s be the center of S, let T' be the set of leaves of S, and let $d = |T'|$ be the number of leaves of S. For each terminal $t_i \in T'$, let s_i be the neighbor of s on the $t_i s$-path (possibly $s_i = t_i$), let $\hat{c}_i$ be the sum of the costs of the edges on the $t_i s$-path, and let $c_i = c(s_i s)$, $i = 1, \ldots, d$. Assume w.l.o.g. that the leaves in T are ordered such that $\hat{c}_1 \le \hat{c}_2 \le \cdots \le \hat{c}_d$. Note that $\hat{c}_i \ge c_i$, and thus $\hat{c}_i \ge \hat{c}_j \ge c_j$ for any $i \ge j$, and that the power of the spider is

$$p(S) = \sum_{i=1}^{d} \hat{c}_i + \max_{1 \le i \le d} c_i$$

We now define our 2-decomposition $\mathcal{D}$. In the case of d even we just take consecutive disjoint pairs $T_i = \{t_{2i-1}, t_{2i}\}$, $i = 1, \ldots, \lfloor d/2 \rfloor$. In the case of d odd, we take the two pairs $\{t_1, t_2\}, \{t_1, t_3\}$ and then add to them the remaining $\lfloor d/2 \rfloor - 1$ consecutive disjoint pairs of the (possibly empty) sequence formed by the remaining terminals in $T \setminus \{t_1, t_2, t_3\}$. Recall that the power of this decomposition is defined to be the sum of the power of its paths, or in other words:

$$p(\mathcal{D}) = \sum_{i=1}^{d} \hat{c}_i + \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\} \qquad \text{if } d \text{ is even}$$

$$p(\mathcal{D}) = \sum_{i=1}^{d} \hat{c}_i + \sum_{i=2}^{\lfloor d/2 \rfloor} \max\{c_{2i}, c_{2i+1}\} + \hat{c}_1 + \max\{c_1, c_2\} + \max\{c_1, c_3\} \quad \text{if } d \text{ is odd}$$

We need to prove that $3p(S) \ge 2p(\mathcal{D})$.

If d is even then we need to prove that:

$$\frac{3}{2}\left(\sum_{i=1}^{d} \hat{c}_i + \max_{1 \le i \le d} c_i\right) \ge \sum_{i=1}^{d} \hat{c}_i + \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\}$$

By rearranging terms we obtain:

$$\sum_{i=1}^{d} \hat{c}_i + 3 \max_{1 \le i \le d} c_i \ge 2 \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\}$$

The latter inequality holds since:

$$\begin{aligned}
\sum_{i=1}^{d} \hat{c}_i + 3 \max_{1 \le i \le d} c_i \quad &\ge \quad \sum_{i=1}^{d/2-1} (\hat{c}_{2i} + \hat{c}_{2i+1}) + 2 \max_{1 \le i \le d} c_i \ge \\
&\ge \quad 2 \sum_{i=1}^{d/2-1} \max\{c_{2i-1}, c_{2i}\} + 2 \max\{c_{d-1}, c_d\} \\
&= \quad 2 \sum_{i=1}^{d/2} \max\{c_{2i-1}, c_{2i}\}
\end{aligned}$$

For the first inequality we applied a standard manipulation of indices, giving up some terms while recalling that all costs and powers are non-negative; the second inequality is since $\hat{c}_{2i+1} \ge \hat{c}_{2i} \ge \max\{c_{2i-1}, c_{2i}\}$; the last equality is obvious.

If d is odd then we need to prove that:

$$\frac{3}{2} \left(\sum_{i=1}^{d} \hat{c}_i + \max_{1 \le i \le d} c_i \right) \ge \sum_{i=1}^{d} \hat{c}_i + \sum_{i=2}^{\lfloor d/2 \rfloor} \max\{c_{2i}, c_{2i+1}\} + \hat{c}_1 + \max\{c_1, c_2\} + \max\{c_1, c_3\}$$

By rearranging terms we obtain that we need:

$$\sum_{i=1}^{d} \hat{c}_i + 3 \max_{1 \le i \le d} c_i \ge 2\hat{c}_1 + 2 \max\{c_1, c_2\} + 2 \max\{c_1, c_3\} + 2 \sum_{i=2}^{\lfloor d/2 \rfloor} \max\{c_{2i}, c_{2i+1}\}.$$

If $d > 3$, the latter inequality holds since:

$$\begin{aligned}
\sum_{i=1}^{d} \hat{c}_i + 3\max_{1\le i\le d} c_i &= \hat{c}_1 + (\hat{c}_2 + \hat{c}_3) + \hat{c}_4 + \hat{c}_d + 3\max_{1\le i\le d} c_i + \sum_{i=3}^{\lfloor d/2\rfloor} (\hat{c}_{2i-1} + \hat{c}_{2i}) \\
&\ge (\hat{c}_1 + \hat{c}_4) + \hat{c}_d + 2\max\{c_1, c_2\} + 3\max_{1\le i\le d} c_i + 2\sum_{i=2}^{\lfloor d/2\rfloor -1} \max\{c_{2i}, c_{2i+1}\} \\
&\ge 2\hat{c}_1 + \max\{c_{d-1}, c_d\} + 2\max\{c_1, c_2\} + \\
&\quad \max\{c_{d-1}, c_d\} + 2\max\{c_1, c_3\} + 2\sum_{i=2}^{\lfloor d/2\rfloor -1} \max\{c_{2i}, c_{2i+1}\} \\
&= 2\hat{c}_1 + 2\max\{c_1, c_2\} + 2\max\{c_1, c_3\} + 2\sum_{i=2}^{\lfloor d/2\rfloor} \max\{c_{2i}, c_{2i+1}\}.
\end{aligned}$$

The first equality follows by applying a standard manipulation of indices; the first inequality is since $\hat{c}_{i+1} \ge \hat{c}_i \ge \max\{c_{i-1}, c_i\}$; the last inequality is since $\hat{c}_4 \ge \hat{c}_1$, $\hat{c}_d \ge \max\limits_{1\le i\le d} c_i \ge \max\{c_i, c_j\}$ for any i, j, and the last equality follows by applying a standard manipulation of indices.

If $d = 3$, we must prove that:

$$\hat{c}_1 + \hat{c}_2 + \hat{c}_3 + 3\max_{1\le i\le 3} c_i \ge 2\hat{c}_1 + 2\max\{c_1, c_2\} + 2\max\{c_1, c_3\},$$

and this follows since $\hat{c}_2 \ge \hat{c}_1$ and $\hat{c}_3 \ge \max\{\hat{c}_1, \hat{c}_3\} \ge 2\max\{c_1, c_3\}$. This completes the proof of the lemma. □

Our algorithm for Min-Power Terminal Backup is as follows.

Algorithm 4: Approx-T-Backup$(G = (V, E), c, T)$ (ratio 3/2)

---

1 **construct** an instance $(G' = (T, E'), c')$ of Min-Cost T-Cover:

- For every $\{t_i, t_j\} \in T$ with $i \ne j$ let L_{ij} be a $t_i t_j$-path of minimum power.
- The graph G' is a complete graph on T with edge costs $c'(t_i t_j) = p(L_{ij})$.

2 **compute** a minimum cost T-cover F' in G', c'.

3 **return** $F = \cup\{L_{ij} : t_i t_j \in F'\}$

We note that the problem of computing a minimum power t_it_j-path can be solved in polynomial time by a simple reduction to its min-cost variant, c.f. [1, 24]. All the other parts of the algorithm can also be implemented in polynomial time. The following statement is used to prove that the approximation ratio of the algorithm is 3/2.

Lemma 5.5.

(i) *If F' is a T-cover in G' then $F = \cup\{L_{ij} : t_it_j \in F'\}$ is a T-backup in G and $p(F) \leq c'(F')$.*

(ii) $\mathsf{opt}' \leq 3\mathsf{opt}/2$, *where* opt' *is the minimum cost of a T-cover in G', c'.*

Proof. F is a T-backup since F' is a T-cover, and since L_{ij} connects t_i and t_j for every $t_it_j \in F'$. Also, $p(F) \leq c'(F')$ since

$$p(F) = p\left(\bigcup_{t_it_j \in F'} L_{ij}\right) \leq \sum_{t_it_j \in F'} p(L_{ij}) = \sum_{t_it_j \in F'} c'(t_it_j) = c(F') \ .$$

We now prove that $\mathsf{opt}' \leq 3\mathsf{opt}/2$. Let F be an optimal inclusion minimal solution to Min-Power Terminal Backup in (G, c, T), so $p(F) = \mathsf{opt}$. By Lemma 5.4 there exists a 2-decomposition $\mathcal{D}$ of F with $p(\mathcal{D}) \leq 3p(F)/2 = 3\mathsf{opt}/2$. To every path $L_{ij} \in \mathcal{D}$ corresponds an edge $e_{ij} = t_it_j$ in G' and $c'(e_{ij}) \leq p(L_{ij})$. Let $F' = \{e_{ij} : L_{ij} \in \mathcal{D}\}$. Then F' is a T-cover in G', since e_{ij} and L_{ij} have the same endnodes t_i, t_j, and since F is a T-cover. Hence $\mathsf{opt}' \leq c'(F')$. Thus:

$$\mathsf{opt}' \leq c'(F') = \sum_{e' \in F'} c'(e') \leq \sum_{i,j} p(L_{ij}) = p(\mathcal{D}) \leq 3p(F)/2 = 3\mathsf{opt}/2 \ .$$

$\square$

Theorem 1.5 now easily follows from Lemma 5.5. Let F, F' be as in the algorithm. Then, by Lemma 5.5, we have $p(F) \leq c'(F') = \mathsf{opt}' \leq 3\mathsf{opt}/2$.

The proof of Theorem 1.5 is complete.

6. Removing the need of k-restricted approach

We can obtain T-covers as in Theorem 1.3 without using k-restricted covers. This improves the running time of the main approximation algorithm.

The "hypergraphic" linear program $LP(T)$ below has a variable x_R for every star R and it is a relaxation for the problem of finding a T-cover of minimum power. Let $\mathcal{R}$ be the collection of all stars of the input graph.

$$\begin{array}{lll} \min & \displaystyle\sum_{R\in\mathcal{R}} p(R)x_R & \\ \text{s.t.} & \displaystyle\sum_{R\in\mathcal{R}, v\in V(R)} x_R \geq 1 & \forall v \in T \\ & x_R \geq 0 & \forall R \in \mathcal{R} \end{array}$$

The approximation algorithm is the same as in Theorem 1.3, but it uses $LP(T)$ instead of $LP_k(T)$. The remaining challenge is solving $LP(T)$, which has exponentially many variables.

For this, we use an auxiliary linear program $ALP(T)$, which has variables $y_{(v,u)}$ for all 2-tuples (v,u) with $v,u \in V$ with $v \neq u$ and $uv \in E(G)$, and $z_{(w,v,u)}$ for all 3-tuples (w,v,u)) with $w,v,u \in V$, $u \notin \{v,w\}$ and $c(wu) \leq c(vu)$ (note that $w = v$ is possible). $ALP(T)$ is:

$$\begin{array}{lll} \min & \displaystyle\sum_{(v,u)} c(uv)y_{(v.u)} + \sum_{(w,v,u)} c(wu)z_{(w,v,u)} & \\ \text{s.t.} & \displaystyle\sum_{v} y_{(v.u)} + \sum_{(w,v)\ \mid\ c(uv)\leq c(wv)} z_{(u,w,v)} \geq 1 & \forall u \in T \\ & z_{(w,v,u)} \leq y_{(v,u)} & \forall (w,v,u) \\ & z_{(v,v,u)} = y_{(v,u)} & \forall (v,u), v \neq u \\ & y_{(v,u)} \geq 0 & \forall (v,u), v \neq u \\ & z_{(w,v,u)} \geq 0 & \forall (w,v,u), v \neq u \text{ and } c(wu) \leq c(vu) \end{array}$$

Let us check this equivalence. If we have a solution to $LP(T)$, from variables x_R we obtain the variables $y_{(v,u)}$ and $z_{(w,v,u)}$ in $ALP(T)$ as follows: we start with each such variable as 0, and for every R, star with center u and leaves $v_1, v_2, \ldots, v_k$ arranged in non-increasing order of costs $c(uv_1) \geq c(uv_2) \geq \cdots \geq c(uv_k)$, we add x_R to $y_{(v_1,u)}$ and to $z_{(v_j,v_1,u)}$, for $j = 1$ to k. Notice that $p(R) = c(uv_1) + \sum_{j=1}^{k} c(uv_j)$, which is at most the increase in the objective function of $ALP(T)$. Also, notice that $z_{(v,v,u)} = y_{(v,u)}$ for all $v \neq u$ since every time $y_{(v,u)}$ is increased, $z_{(v,v,u)}$ is increased by the same amount.

Moreover, the constraint, for a given $u \in T$, $\sum_v y_{(v.u)} + \sum_{(w,v)} z_{(u,w,v)} \geq 1$ is satisfied, as for every star R with $u \in V(R)$, x_R contributes to either $y_{(v_1,u)}$ (when u is the center of R), or $z_{(u,w,v)}$ (when v the center of R and w the first child of R), and using $\sum_{R \in \mathcal{R}, v \in V(R)} x_R \geq 1$.

Now suppose we have a feasible solution to $ALP(T)$. One by one, go through all 2-tuples (v,u) with $v \neq u$. Let $w_1, w_2, \ldots, w_q$ be the nodes with $c(w_j u) \leq c(vu)$ sorted such in non-decreasing order of $z_{(w_i,v,u)}$. As for all i, we have that $z_{(w_i,v,u)} \leq y_{(v,u)} = z_{(v,v,u)}$, we may assume that $v = w_q$. For $i = 1, 2, \ldots, q$, star $R_i = R_i(v,u)$ will have center u and children $w_i, \ldots, w_q$.Set $x_{R_1} = z_{(w_1,v,u)}$, and for $i > 1$ set $x_{R_i} = z_{(w_i,v,u)} - z_{(w_{i-1},v,u)}$. Using that $c(w_j u) \leq c(vu)$ for all u, we have that $p(R_i) \leq c(vu) + \sum_{j=i}^{q} c(w_i u)$. Using that $y_{(v,u)} = z_{(w_q,v,u)}$, we deduce that:

$$\begin{aligned}
\sum_{i=1}^{q} p(R_i) x_{R_i} &\leq z_{(w_1,v,u)} \left(c(vu) + \sum_{j=1}^{q} c(w_i u) \right) + \\
&\quad \sum_{i=2}^{q} \left(z_{(w_i,v,u)} - z_{(w_{i-1},v,u)} \right) \left(c(vu) + \sum_{j=i}^{q} c(w_i u) \right) \\
&= c(vu) z_{(w_q,v,u)} + \sum_{i=1}^{q} z_{(w_i,v,u)} c(w_i u) \\
&= c(vu) y_{(v,u)} + \sum_{i=1}^{q} z_{(w_i,v,u)} c(w_i u).
\end{aligned}$$

Therefore, while doing this for all 2-tuples (v,u) with $v \neq u$, we obtain a solution of $LP(T)$ without increasing the costs.

Moreover, when creating these stars R_i, we have that

$$\sum_{i, u \in V(R_i)} x_{R_i} = y_{(v,u)}$$

and

$$\sum_{i, w_j \in V(R_i)} x_{R_i} = z_{(w_j,v,u)}.$$

Therefore, for any u, we have $\sum_{R \in \mathcal{R}, u \in V(R)} x_R \geq \sum_v y_{(v,u)} + \sum_{(w,v)} z_{(u,w,v)} \geq 1$, and thus our $LP(T)$ solution is feasible.

Thus, with $LP(T)$ we eliminated the need for k-restricted covers; by comparisons for the Min-Cost Steiner Tree problem, so far, k-restricted decompositions are still needed for the best ratios, or for any linear program with ratio provable better than 2.

We note that it is quite possible that solving one $LP(T)$ is enough (when applying iterative rounding, avoid resolving new linear programs), as it was shown for Min-Cost Steiner Tree by [14].

We also believe that $LP(T)$ has integrality ratio better than $3/2$, but with the methods of this paper we were not able to obtain the ratio of Theorem 1.3, the general case, with respect to $LP(T)$ ($LP(T)$ may be used to bound the output of the iterative rounding algorithm, but the KN-Algorithm's output is directly compared to the optimum).

Acknowledgment:. Gruia and Zeev thank Neil Olver for many useful discussions. Also, Gruia and Zeev acknowledge the support of the Hausdorff Trimester Program for Combinatorial Optimization (held at the Hausdorff Research Institute for Mathematics, University of Bonn).

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