Exercises III

Remarks: In all the algorithms, always explain their correctness and analyze their complexity. The complexity should be as small as possible. A correct algorithm with large complexity, may not get full credit.

- **Question 1:** Consider a directed graph $G(V,E)$ with $V = \{1,2,\ldots,n\}$. Let $A$ be a matrix so that $a_{i,j} = 1$ if there is a (directed) edge $i \to j$.

  A walk from $i$ to $j$ is a non-necessarily simple path from $i$ to $j$ (edges and vertices can appear many times in the path). Show that the $i,j$ entry of $A^k$ ($A^2 = A \cdot A$ and $A^3 = A \cdot A \cdot A$ etc) is the number of walks from $i$ to $j$ in $G$ of length exactly $k$.

  **answer:** TO BE GIVEN

- **Question 2:** A DAG (directed acyclic graph) is a directed graph with no directed cycles (recall: a cycle is a directed path that starts and ends at the same vertex).

  Show that on a DAG the longest path problem is polynomial time solvable. The longest path problem gets a graph as input and requests the longest *simple* path in the graph.

- **Question 3:** suppose we want to chose a permutation (ordering on a line) of $1,2,\ldots,n$. Explain how to create a random permutation in polynomial time. Namely, in the sample space there will be the $n!$ possible orderings, each one of them with probability $1/n!$. Prove your answer.

- **Question 4:** Let $A \subseteq \{1,2,\ldots,n\}$. Say that $f$ is a random permutation of $\{1,2,\ldots,n\}$. We say that $f$ keeps the order of $A$ if for every $i,j \in A$, $i < j$ will imply $f(i) < f(j)$.

  For example: $\{1,2,3,4,5\}$, $A = \{2,4,5\}$. The permutation $\{3,2,1,4,5\}$ keeps the order of $A$ but $\{1,4,2,3,5\}$ does not because 4 comes before 2. Let $|A| = k$. Show that the probability that $f$ keeps the order of $A$ is $1/k!$.

- **Question 5:** Let $G(V,E)$, $V = \{1,2,\ldots,n\}$ be a directed graph (not a DAG). Let $f$ be a permutation of the vertices $\{1,2,\ldots,n\}$. Say that we remove all edges $i,j$ so that $(i,j) \in E$, $f(i) > f(j)$. Let $G'(V,E')$ be the resulting graph.

  1. Show that $G'$ is a DAG.

  2. Show that an edge $(i,j) \in E$ if and only if $f(i) < f(j)$.

  3. Say that $G$ had a simple path $v_1 \to v_2 \ldots v_{k-1} \to v_k$ of length (number of edges) $k-1$. Show that the path also exists in $G'$ if and only if $G'$ keeps the order of $G$.

  4. Show that the probability that the path exists in $G'$ is $1/k!$.

- **Question 6:**

  1. Show that if we run the following algorithm $k! \cdot \ln n$ times with probability at least $1/n$, if a path of length $k$ exists in $G$, we are going to find it.
(a) Randomly permute $1, 2, \ldots, n$ by a random permutation $f$

(b) Remove edges $i \rightarrow j$ so that $f(i) > f(j)$.

(c) Run the algorithm of question 1 and check if the graph $G'$ has a path of length $k$

2. Show that the above algorithm runs in polynomial time if $k$ is constant

3. Show that the algorithm runs in polynomial time if $k = O(\log n / \log \log n)$